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MATHEMATICS FOR ELEMENTARY SCHOOL TEACHERS.

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PRODUCED AS PART OF A PROJECT FOR THE IMPROVEMENT OF CLASSROOM TEACHING, THIS BOOK DISCUSSES THE BASIC CONCEPTS AND OPERATIONS OF ARITHMETIC WHICH ARE TAUGHT TO ELEMENTARY SCHOOL STUDENTS. THE CONCEPT OF A SET IS USED THROUGHOUT AS A BASIS FOR EXPLANATION, AND THE DISCUSSION STRESSES WAYS BY WHICH STUDENTS CAN BE BROUGHT TO UNDERSTAND THE CONCEPTS. THE FIRST CHAPTER COVERS BEGINNING NUMBER CONCEPTS--SETS, PAIRING, COUNTING. SUCCEEDING CHAPTERS COVER THE DECIMAL NUMERATION SYSTEM, ADDITION, MULTIPLICATION, SUBTRACTION, DIVISION, ALGORITHMS FOR ARITHMETIC COMPUTATIONS, AND THE WHOLE NUMBER SYSTEM. THE FINAL SECTIONS OF THE BOOK INCLUDE ANSWERS TO THE EXERCISES GIVEN IN THE CHAPTERS AND DESCRIPTIVE DEFINITIONS OF TERMS USED. THIS DOCUMENT IS AVAILABLE FROM THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS, 1201 SIXTEENTH STREET, N.W., WASHINGTON, D.C. 20036. (DR)

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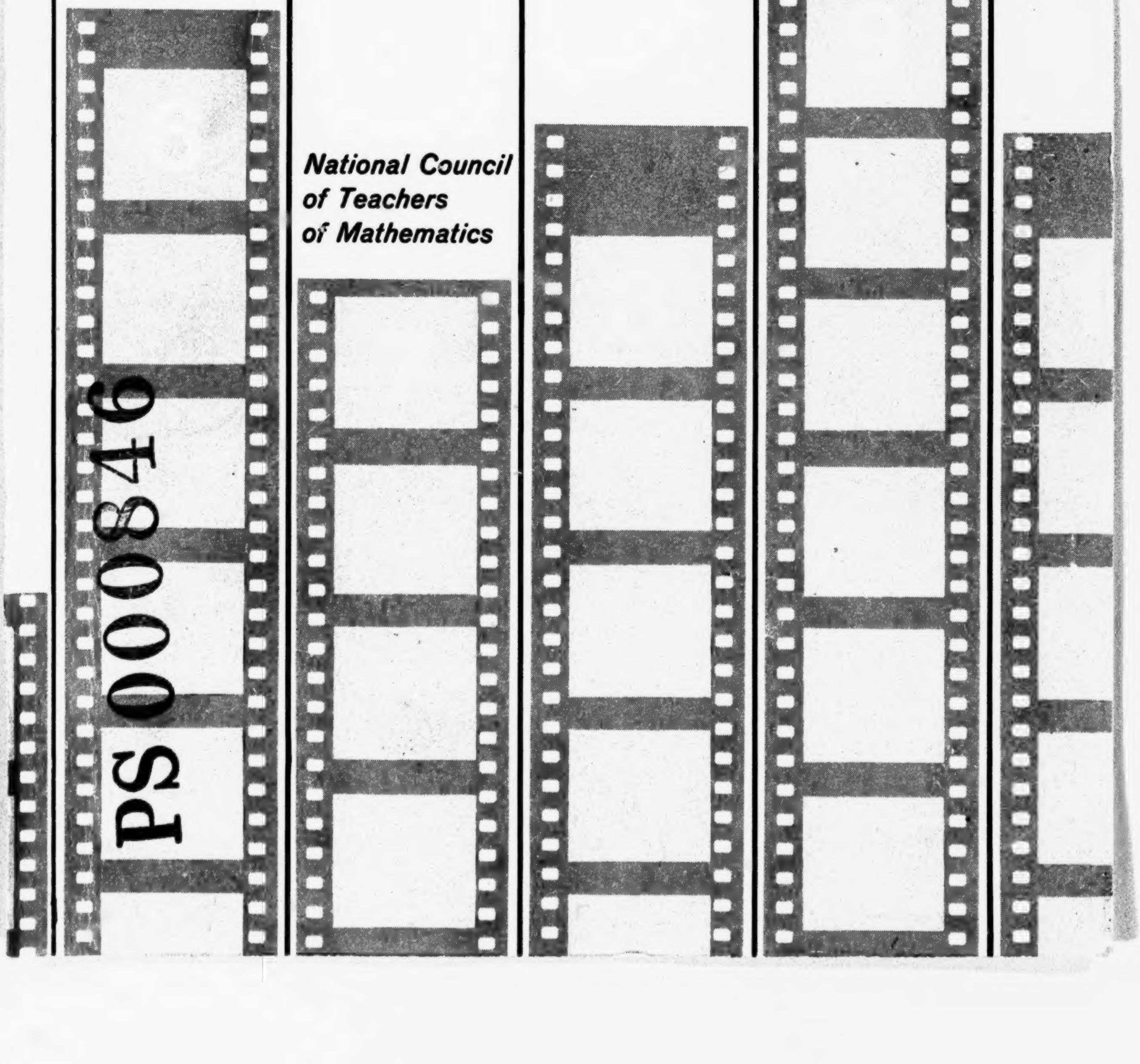

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HISTORY AND ACKNOWLEDGMENTS

This book, although complete in itself, is coordinated with a film series bearing the same title. Together, they represent a project of the National Council of Teachers of Mathematics (NCTM) as it experiments with instructional media for the continual improvement of classroom teaching. The history of the project spans several years.

Early in 1962 Frank B. Allen, president of the NCTM, appointed a Films and Television Committee, composed of Harry D. Ruderman, *chairman*; Emil J. Berger; Joseph A. Raab; and Leonard Simon.

In June of that year the Mathematical Association of America, the School Mathematics Study Group, and the NCTM participated in the Joint Conference on Films and Television in Chicago. Upon the recommendation of the conference that the NCTM undertake production of a series of films for the in-service training of elementary school teachers, President Allen instructed the NCTM Films and Television Committee to prepare a proposal to the National Science Foundation for funds to implement this recommendation.

To assist the committee in preparing the proposal, President Allen appointed also an Advisory Panel, composed of Julius H. Hlavaty, *chairman*; Frank B. Allen; E. Glenadine Gibb; William T. Guy, Jr.; John J. Kelley; John L. Kelley; Helen A. Schneider; and Henry Van Engen.

The Committee and the Panel, with the additional assistance and advice of Max Beberman, Arnold M. Chandler, Don R. Lichtenberg, and Tina Thoburn, prepared the requested proposal in the fall of 1962.

The NCTM Board of Directors accepted the proposal and submitted it to the National Science Foundation (NSF). Pending the decision of the NSF, the Board of Directors, in December, appropriated some NCTM funds for experimentation with the production of films.

Harry D. Ruderman, chairman of the department of mathematics of Hunter College High School, New York, was persuaded to accept the post of executive director for the project. Joseph Moray, of the State College of San Francisco, was selected as film teacher; and Reinald Werrenrath, Jr., of Highland Park, Illinois, was retained as technical consultant.

During the summer of 1963 five experimental films were produced with NCTM funds. By the fall of 1963 the request for funding by the NSF was granted.

The Advisory Panel (enlarged and strengthened by the recruitment of Truman A. Botts, Leon A. Henkin, and Peter J. Hilton) appraised the experimental films, with the active cooperation of the NCTM Board of Directors and several hundred classroom teachers. Making use of the experience thus gained, the Panel went on to plan production of a totally new series on the whole-number system. At every stage in the development of the project, the Advisory Panel was kept informed and [redacted] both critically and constructively.

During the summer of 1964 a writing group prepared text materials for the projected series of films. The group consisted of Harry D. Ruderman, *chairman*; Julia E. Adkins; Don R. Lichtenberg; Leonard Simon; and Robert Spaeth. The materials produced by this group were not only the first drafts of the text materials here published but also the source for the scripts developed for the films. For the major editing and rewriting of these materials, Abraham M. Glicksman and Harry D. Ruderman are responsible.

In the fall of 1964 Davidson Films was selected to produce the film series. John M. Davidson was responsible for the ultimate production of the film series. The insight that characterized his advice in development of the scripts was invaluable.

During the actual production year (1964-65) many people contributed of their time, energy, and advice. Besides those already named were the following:

Members of the NCTM Films and Television Committee: Robert C. Clary, Wilma Rollins, Don R. Lichtenberg, and David W. Wells

Successive chairmen of the NCTM Committee on Instructional Media: H. Vernon Price and Robert R. Willson

Other members of the Committee on Instructional Media: James F. Gray, H. Stewart Moredock, Thomas A. Romberg, and George S. Cunningham

Executive directors of the Committee on Educational Media of the Mathematical Association of America: A.N. Feldzamen and Philip E. Miles

Myrl H. Ahrendt, of NASA, formerly executive secretary of the NCTM, who helped prepare the original proposal to the NSF and worked on the project in its early stages

James D. Gates, executive secretary of the NCTM

Bruce E. Meserve, president of the NCTM, 1964-66

Donovan A. Johnson, president of the NCTM, 1966-68

Grateful acknowledgment is here expressed to Sonia Collins and Nancy Leist for many hours of typing the manuscript for this book.

For the detailed and dedicated work that made ultimate publication possible, a grateful word of thanks is extended to Editorial Associate Julia A. Lacy and her assistants, Elizabeth G. Emanuel, Dorothy C. Hardy, and Gladys F. Rose, who have worked under the direction of Charles R. Hucka, assistant executive secretary of the NCTM.

JULIUS H. HLAVATY
Chairman, Advisory Panel
Films in Mathematics for Elementary School Teachers

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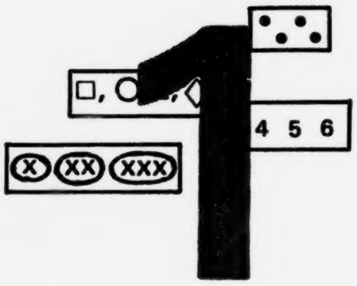
MATHEMATICS FOR ELEMENTARY SCHOOL TEACHERS

Note to the Reader

You will find that sets of exercises appear in every chapter, and that answers to all of them appear in the back of the book. Before consulting the answers you will want to do your own figuring. For your convenience, working space has been provided beneath each exercise as needed.

BEGINNING NUMBER CONCEPTS

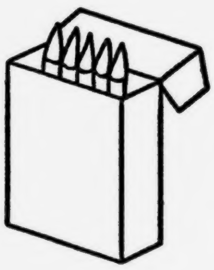


1. **What is a set?**
2. **What is a one-to-one correspondence?**
3. **When are two sets equivalent?**
4. **What basic ideas are involved in counting?**

Cover these dots ● ● ● ● ● ● ● ● ● ● ● ● ● with your hand. When you finish reading this sentence, uncover the dots for just a second or two and try to decide how many dots are under your hand. How many dots are there?

Were you able to determine the number of dots from a quick glance? Did you try to count them from a mental picture? Did you try to group them?

In the early grades children need help in developing an ability to count. For example, to find the number of crayons in this set of crayons, one child might "count":



"One, two, three, five, ten."

Another might rapidly chant:

"One, two, three, four, five, six."

The first child skipped over certain numbers in his attempt at counting. The second child failed to pair each number with a crayon.

In this chapter we shall explore the basic ideas of set, matching, number, order, and counting. These ideas can provide children with a strong foundation for building an understanding of mathematics.

set **matching** **number** **order** **counting**

MATCHING SETS

How can we decide whether or not there are enough seats in your classroom for your pupils without knowing the actual number of pupils or the number of seats in the room?

Inherent in questions like this is the idea of *pairing* objects. We can try to pair each pupil with a seat. If we succeed in placing each child in a seat, then there are certainly enough seats for the children. If we run short of seats before every child is seated, then there are more children than seats. If there are seats left over, then there are fewer children than seats. When seats and children "match," there are just as many of the one as of the other.

Primitive people used the ideas of pairing objects and matching sets. A hunter might report that he saw as many men as a dog has legs. Another might report that he caught as many fish as he has eyes. Notice that each reported in terms of model sets that were familiar or known to the person hearing the report.

Young children discover the idea of matching early. They learn to match their feet with a pair of shoes, the fingers of the left hand with the fingers of the right hand, the cups and saucers on a table, and so on. They also discover that certain pairs of sets do not match: "Jim has more blocks than Mary." They recognize that if two sets cannot be matched, one of these sets has more members than the other.

Before we look more carefully at matching, let us look at the idea of a *set* of things. In mathematics, the term "set" is used to mean a collection or aggregation or group of objects or ideas. For example, a set of tools might consist of a hammer, a screwdriver, and a saw. The hammer, the screwdriver, and the saw are called *members* or *elements* of this set of tools. A desk set might consist of a pen, a pencil, a penholder, a blotter, and a letter opener. A set might consist of the colors red, orange, and yellow. Another set might consist of the number 3 and the color blue.

One way to specify a set is shown below:

$$\{\text{Henry, John, Bill}\}$$

This is read: "The set whose members are Henry, John, and Bill." The names of the members of the set are enclosed in braces and separated by commas.

For convenience, the set may be designated by a single symbol, frequently by a capital letter.

$$A = \{\text{John, Henry, Bill}\}.$$

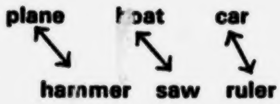
We read this: "*A* is the set whose members are Henry, John, and Bill."

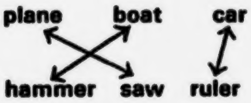
$B = \{\text{chair, moon, a, doll}\}.$

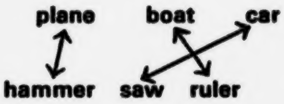
$C = \{\text{plane, boat, car}\}.$
$D = \{\text{hammer, saw, ruler}\}.$

The members of a set need not be related to each other in any special way (for example, see set *B*). However, we should be able to tell whether a given object belongs to the set or does not belong to the set.

Now let us consider two other sets. *C* is the set consisting of a plane, a boat, and a car. *D* is the set consisting of a hammer, a saw, and a ruler. How can we pair the members of these two sets? Below we see three ways to pair the members; the double arrows show the pairing.







Each way of pairing shows that the two sets match. There are also other ways to match these two sets.

Notice that, in any one of these ways of pairing the members of the two sets, each member of set *C* was paired with a member of set *D*, and each member of set *D* was paired with a member of set *C*. In no case are two members of either set paired with the same member of the other set. Children might say that the sets match because there is "none left over," meaning that no member of either set is left without a "partner" in the other set.

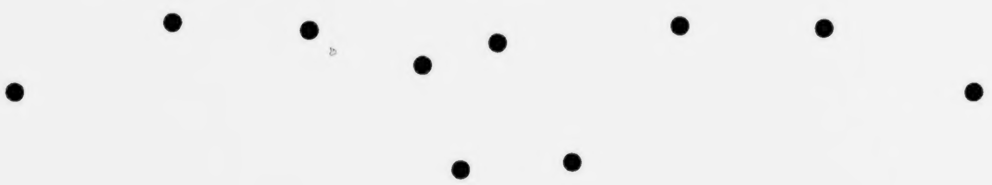
> *When we have such a pairing between the elements of two sets, we say that a one-to-one correspondence has been established between the two sets.*

Early elementary school children need not use the language "one-to-one correspondence," but they should be able to recognize when two sets "match" and when they do not match. Both of these situations are important. However, at this stage we shall concentrate on the case where sets match.

➡NOTE.—*In the exercise sets that follow, both in this chapter and throughout the book, working space is provided after each exercise for your convenience in figuring.* (*Answers to all exercises appear at the back of the book.*)

Exercise Set 1

1. Tell how to decide without counting whether or not there are as many dots below as there are fingers on two hands.



2. Suppose you do not know how many chairs there are in the school library, nor how many people there are on the school staff. How could you decide in advance, *without counting*, whether or not there are enough chairs for the staff to hold a meeting in the library?

3. Directing your attention again to Exercise 2, name the three possibilities that exist when comparing the set of people on the school staff and the set of chairs in the school library.

4. Write the names of the days of a week within a pair of braces. Name the set thus indicated with a capital letter, A.

5. Display one pairing of the elements of set B and the elements of set A from Exercise 4.

$B = \{a, b, c, d, e, f, g\}$.

6. When can we say that a one-to-one correspondence has been established between two sets?

7. Below are five sets. For which pairs of these sets can we establish a one-to-one correspondence?

$A = \{a, b, c\}$.
$B = \{$chair, comb, Mary, Jane$\}$.
$C = \{$orange$\}$.
$D = \{$dog, sun, Bill, eraser$\}$.
$E = \{$pencil, pen, book$\}$.

8. For sets *A* and *E* in Exercise 7, show six ways to pair the elements. (If you are ever asked to produce all possible matchings of your fingers and your toes, don't do it. There are 3,628,800 matchings!)

NUMBER

We have considered the idea of a set as a collection of things which may be physical objects or abstract ideas. We have also considered the notion of pairing the members of sets. Now suppose that at the beginning of a school year you were given a set of class record cards. You might pair a record card from the set of record cards with a student from the set of students. If each record card were paired with a pupil and each pupil were paired with a record card, we could say the two sets have a one-to-one correspondence; that is, they *match*.

Now let us see how the idea of matching is related to the idea of number.

Early man used a matching process. For example, he might have made a tally mark, or cut a notch in a piece of wood, or gathered a pebble for each animal he saw. He then told others that he saw *as many* animals as the marks or pebbles he displayed.

Young children also use the idea of matching. In a typical situation two boys might notice that their toy airplanes match because they can pair them "one-for-one." This primitive notion of one-to-one correspondence, or matching, leads to the idea of "as many as," and this in turn leads to the idea of "number." Let us see how this development comes about.

A = {John, Peter, hat}.
B = {comb, dish, pencil}.
C = {shoe, chain, orange}.

The sets *A* and *B* indicated here can be matched. Sets *A* and *C* can also be matched, as can sets *B* and *C*.

Two sets, such as *A* and *C*, whose members can be paired in a one-to-one correspondence are said to be *equivalent*. (This is not the same as saying the sets are *equal*. Equal sets have exactly the same members. Equivalent sets can have different members.) Imagine other sets that are equivalent to set *A*, to set *B*, and to set *C*—for example, {hammer, moon, tree}.

A collection consisting of the sets *A*, *B*, and *C* above is an example of what we shall call a *family* of sets. In general, we shall call a collection of sets in which any two sets are equivalent a *family*. Any two sets in such a collection can be matched in a one-to-one correspondence.

If two sets are not equivalent, they cannot be in the same family. For example, consider the set

$$D = \{\text{square, triangle, circle, diamond}\}.$$

Could set D be in the same family with sets A, B, and C above? Clearly not, because D is not equivalent to A, B, or C.

Suppose we were to form a family of sets including set D as a member. All the sets in this new family would be equivalent to set D. None of them would be equivalent to sets A, B, or C above.

Consider the set of lakes known as the Great Lakes. Can this set belong in a family together with set D? Why, or why not?

$E = \{\text{window, door}\}.$

$F = \{\text{hand, swing}\}.$

$G = \{\text{pen, pencil}\}.$

Sets E, F, and G *can* form a family of sets. Why? Give some other examples of sets that can belong in a family that also contains E, F, and G as members.

To summarize: From our definition of a family it follows that every two sets in the same family are equivalent. This means that if we select any two sets from the same family, the elements of the sets can be paired one-to-one.

Now we are ready to think of *number*. Consider a family that contains set E above. Unofficially we feel that each set in this family has two members. "Twoness" is a common property (characteristic) shared by all the sets in this family. Similarly, imagine a family of sets where one of the sets is the set of fingers on one hand. The set of lakes called the Great Lakes is another eligible member of this same family. "Fiveness" is a common property of these sets. When a child recognizes which sets are eligible for membership in this family, he is beginning to form an idea of the number "five."

To communicate ideas, we use words or symbols. We have special words and symbols for talking about numbers. To convey the number idea or common property of a family of sets that includes the set of Great Lakes, we use the words "five," or *cinq* (French), or *pyet* (Russian), or others. The words describing this number idea are different, but they refer to only one number idea. Of course, we can write the symbols "5," "V," "五," or other symbols to refer to the number of elements in the set of Great Lakes. A symbol such as "5" is not the *number* five; it is simply one way to name the number five. We shall see that there are many other ways.

In order to denote numbers, we use symbols called *numerals*. Thus

"5" or "five" are numerals used to convey the idea of the common property of a particular family of sets. When we write the word "England," we are writing the name of a country. Similarly, when we write the numeral "5," we are writing a name for a number. We can change the number symbol "5" to "V," but we do not change the number idea in our minds.

As children learn to recognize the number property of a set of elements such as set A, they learn to associate the number 4 with this set.

$A = \{\text{comb, book, clock, apple}\}$.

To indicate that 4 is associated with set A, we have a standard symbolism. To show that set A has 4 elements, we can write "$n(A) = 4$," which may be read in any of the following ways:

The number of elements in set A is four.
The number property of set A is four.
The number associated with set A is four.
The number of set A is four.

We shall use the symbol $n(A)$ for our discussion.

Exercise Set 2

1. How can you decide whether or not two sets can belong to the same family?

2. What common property do any two sets in a family of sets have?

3. When we write $n(D) = 7$, how many members are we asserting there are in set D?

4. Give an illustration of a set D such that $n(D) = 7$.

5. List the elements of another set that can be in the same family of sets as set D of Exercise 4.

6. What does $n(D) = n(E)$ mean?

7. What number is associated with any set in a family of sets to which D of Exercise 3 belongs?

ORDER

We have seen how matching is involved in developing the idea of a family of equivalent sets and how the idea of number as a common property emerges for each family. Numerals, or number names, were invented to communicate number ideas. Our next task is to determine a suitable family for any given set.

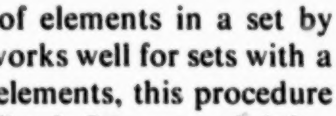
Imagine that we have set up some families of sets. We shall refer to each family by the number property of that family. For example:

FAMILY 1 — a collection of sets, each in one-to-one correspondence with the set of tallies /

FAMILY 2 — a collection of sets, each in one-to-one correspondence with the set of tallies //

FAMILY 4 — a collection of sets, each in one-to-one correspondence with the set of tallies ////

FAMILY 9 — a collection of sets, each in one-to-one correspondence with the set of tallies ///// ////



Now to decide the number of elements in set A, it is only necessary to decide to which family set A can belong. To do this, a child might match set A with a set from one of the above families whose number is already known. When, for example, he finds a set that can be matched (placed in one-to-one correspondence) with set A, then set A can belong to the same family as that set. He will find that set A matches a set from Family 4. Therefore, set A can belong to Family 4 because the number of elements in set A is 4.

This procedure of determining the number of elements in a set by matching it with a set from a particular family works well for sets with a small number of elements. If a set contains 173 elements, this procedure of matching is cumbersome and needs to be refined. One way of doing this is by counting. We have grown so accustomed to counting that we count automatically. Nevertheless, let us examine counting a little more closely.

Counting depends on the fact that there is a natural order among numbers. This natural order stems from the fact that often sets cannot be made to match.

$A = \{\text{Ed, Richard, Bill, Ralph}\}$.

$B = \{\text{Pat, Irene, Sonia}\}$.

$C = \{\text{Ed, Richard, Bill}\}$.

Suppose that a child does not know that set *A* has more elements than set *B*. If he tries to pair the boys from set *A* with the girls from set *B*, he soon realizes that these sets cannot be made to match. No matter how he tries to pair the members of these sets, he finds that a member of set *A* is "left over." Therefore, he orders the sets and says, "There are more boys than girls."

The child's matching of the set of girls with "part" of set *A* gives us a hint as to how to order nonmatching sets such as *A* and *B* above. Now consider set *C*, namely {Ed, Richard, Bill}. Observe that all of its elements are members of set *A*, so that set *C* is actually a "part" of set *A*. But set *C* matches set *B*. Thus, a "part" of set *A* can be matched with all of set *B*. When this happens, we say that *A* has *more* elements than *B* or that the number of set *A* is *greater* than the number of set *B*. We also say that *B* has fewer elements than *A* or that the number of set *B* is *less* than the number of set *A*.

We have said that set *C* is "part" of set *A*. Because every member of set *C* is also a member of set *A*, we say that set *C* is a *subset* of set *A*. In general, if every member of set *X* is also a member of set *Y*, then it is customary to say that *X* is a *subset* of *Y*. Notice that, according to this definition, every set *X* is a subset of *itself* (because every member of *X* is surely a member of *X*!). Returning to the above example, where set *C* is a subset of set *A*, observe that set *A* is not a subset of set *C*. Notice that there is a member of set *A* (namely, Ralph) which is not a member of the subset *C*. Because of this we also call *C* a *proper subset* of *A*. In general, when set *X* is a subset of set *Y* and there is an element in set *Y* that is not in set *X*, then we say that set *X* is a *proper subset* of set *Y*.

$D = \{\square, \Delta, \text{Henry, banana}\}$.

$E = \{\text{chair, table}\}$.

As another example, consider the sets *D* and *E*. To show that the number of set *E* is less than the number of set *D*, we must show that set *E* can be placed in a one-to-one correspondence with a "part" or proper subset of set *D*.

This is easy to do. Consider a proper subset of *D* consisting of Henry and the banana. This proper subset of *D*—that is, {Henry, banana} —can be matched one-to-one with set *E*. (The pairing can be achieved by thinking of Henry on the chair and the banana on the table.) Therefore the number of set *E* is less than the number of set *D*. It can be shown that the number of any set equivalent to *E* is less than the number of any set equivalent to *D*.

The number of set D is 4. The number of set E is 2. To indicate that 2 is less than 4, we write "$2 < 4$," which is read: "Two is less than four."

We can show that set F is equivalent to a proper subset of set G. (For example, pair dog with tree!) Therefore, 1 is less than 5. We write "$1 < 5$," which is read: "One is less than five."

$F = \{\text{dog}\}$.
$G = \{\text{plane, train, ship, rug, tree}\}$.

So we can use the matching of sets to decide upon the order of numbers. For example, we have just seen that 2 is less than 4 and 1 is less than 5. In the same manner we can establish that

$$1 < 2,\ 2 < 3,\ 3 < 4,\ 4 < 5,\ 5 < 6,$$

and so on.

Thus, using the basic idea of matching sets and subsets, the teacher can structure children's experiences so as to develop the ideas of number and order among numbers. In doing this, the teacher may prefer to use informal language.

Exercise Set 3

1. Which of the sets indicated below are subsets of set G? $G = \{\text{block, crayon, paper}\}$.

$A = \{\text{block}\}$.
$B = \{\text{crayon}\}$.
$C = \{\text{paper}\}$.
$D = \{\text{block, crayon}\}$.
$E = \{\text{toy, crayon}\}$.
$F = \{\text{block, crayon, paper}\}$.

2. Which of the sets in Exercise 1 are proper subsets of set G?

3. Show that set A can be matched one-to-one with a subset of set B.

$A = \{\text{Mary, John}\}$.
$B = \{\text{black, pencil, pen}\}$.

4. If, in Exercise 3, set A can be matched with a proper subset of B, what can be said of the number of set A and the number of set B?

5. How do you read "2 < 3"?

6. What does "$n(A) < n(B)$" mean?

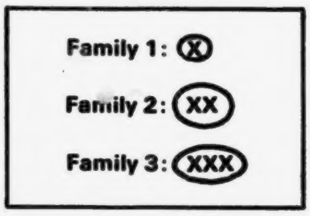
7. Make up two sets (call them "E" and "F") to show that 1 < 2.

8. Make up two sets (call them "R" and "S") to show that $n(R) < n(S)$.

COUNTING

We have established a procedure for ordering numbers. Listed in order, they are 1, 2, 3, 4, 5, 6,..., with the three dots to indicate that the pattern continues without end. The next number may always be obtained by adding 1 to the preceding number. (The notion of adding will be carefully examined in a later unit on addition.)

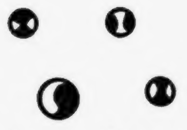
Another way to develop the idea of numbers in order is to begin with a representative set which could be in Family 1. This set contains a single element. Now form a new set consisting of this element together with another element. This new set is in Family 2. In a similar manner, by including still another element, we form a set in Family 3, and so on.

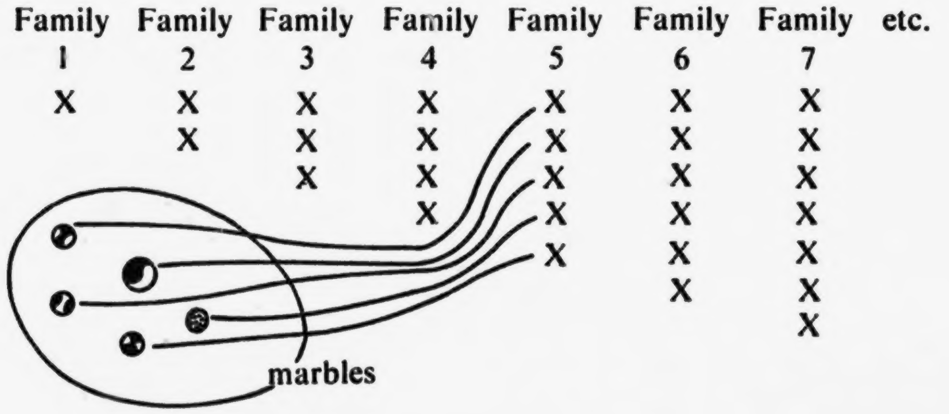


You will recall (see page 8) that to decide the number of elements in a given set, it is only necessary to find a family to which it can belong. It has that family number. This is easily done for sets with a small number of elements. But if a set contains a great many members, and we do not know the number of elements and want to find out, the above procedure needs to be refined. Counting is a refinement of the above procedure.

Now we are ready to make THE BIG STEP. Suppose we want to find the number of some set. We can do this by *pairing the objects of that set with the numbers in the set of ordered numbers.*

For example, here is pictured a set of marbles. Let us consider this the set whose number we are trying to find. If we go back to matching objects, we can select a representative set from a family whose number is known and try to match this set with the marbles as shown below:

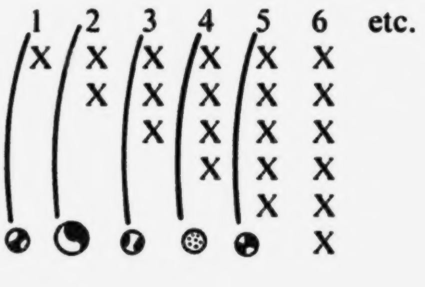




We attempt to pair the marbles with the members of the various representative sets until a one-to-one correspondence is established. In this case, the set of marbles matches a set in the 5 family.

Of course, it is inconvenient to carry around a representative set of X's, or rocks, or sticks, or any other objects. A giant step in man's mathematical development (and also in the child's) was taken when man realized that he could always "carry" the numbers with him.

INSTEAD OF PAIRING WITH OBJECTS, WE PAIR WITH NUMBERS. Let us now find the number of marbles in the set by pairing marbles *with numbers.*



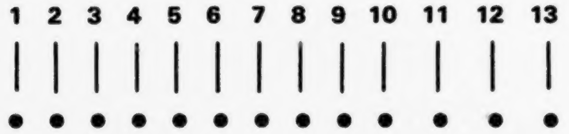
Remember, the numbers are *ordered.* So begin by pairing any marble with 1. Any other is then paired with the next number in order—2.

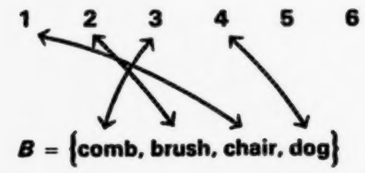
This pairing continues until each marble is paired with a number.

> *The last ordered number paired with a marble is the number that "counts" the set of marbles.*

Therefore, there are five marbles in the set.

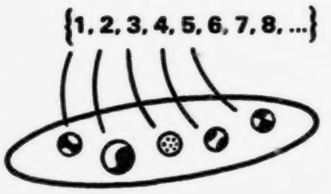
To find how many objects are in a set, you simply count, or pair each object with a number in order. The number paired with the last object to be accounted for is the number of objects in the set. When a child can perform this pairing to find the number of objects in a set, then he has truly learned to count.





It is interesting to note that the members of a set, such as set *B*, may be taken in any order when we count them. For example, the chair may be selected first, but it must then be paired with 1; the next object selected, say the brush, must be paired with the next number in order, 2—and so on. The last number in order with which a member of a set is paired is the number of members of the set.

Why is it that the last number in order with which a member of a set is paired is the number of elements in that set? The reason is to be found in a rather remarkable fact that is often overlooked. Suppose that from the entire set of ordered numbers {1, 2, 3, 4,...} we select any proper subset consisting of consecutive numbers listed in order beginning with 1. For example, we might select {1, 2, 3, 4, 5}. The remarkable fact is that the number of elements in this proper subset is the *last* member of this set when the members are named in order. As another example, think of the subset {1, 2, 3, 4, 5, 6, 7, 8}. Again, the number of elements in the subset is the last member of the subset when the elements are arranged in order.



When we counted the number of marbles in the set, we were actually matching the set of marbles with a certain ordered subset of the ordered set of numbers. We know the number of elements in the subset of numbers. Therefore, the set of marbles can belong to the same family as the set {1, 2, 3, 4, 5} and has the same number property.

However, when we count, we accept this "obvious" fact and simply say that the number of elements in a set is the last number (in our ordered set of numbers) with which a member of the set is paired.

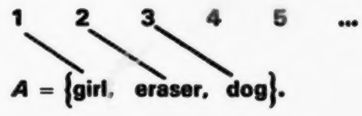
The set of numbers {1, 2, 3, 4, 5,...}, which we have used to count elements of a set, is called the set of *counting numbers*. This set of numbers is also called the set of *natural numbers*.

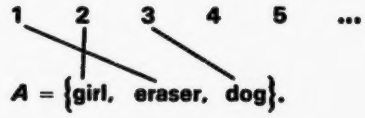
In our discussion of families of sets, we did not mention a certain special family. It is special because this family has only a single set. Think of the set of children over 12 feet tall in your class. How many children are in this set? Think of the set of all U.S. presidents born before 1492. How many elements are in this set? If a set has no elements at all, we call it the empty set and represent it with the symbol "{}." Since any example of the empty set contains the same members (namely, no members!), we say there is only one empty set and call it *the* empty set. The number assigned to the empty set is 0; that is, $n(\{\}) = 0$.

Let us form a new set consisting of 0 together with the counting numbers: {0, 1, 2, 3, 4, 5,...}. This set is called the set of *whole numbers*. (Some mathematicians prefer to call the set {0, 1, 2, 3, 4, 5,...} the set of *natural numbers*. Unfortunately, there is no unanimity on this score.)

Exercise Set 4

1. The members of set *A* can be counted in six different ways. Two of these ways are shown at the right. Find the other four ways.





2. In Exercise 1, for each of the six different ways that you counted the elements in set *A*, what was the last number matched with an element from set *A*?

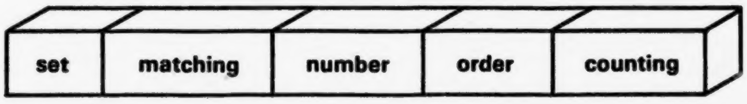
3. How would you describe counting?

4. What is the least number in the set of counting numbers?

5. What is the least number in the set of whole numbers?

6. Give an example of a set that has no members. What do we call such a set?

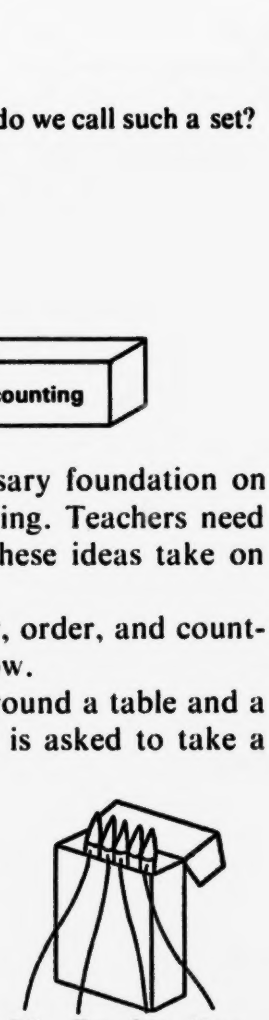
SUMMARY



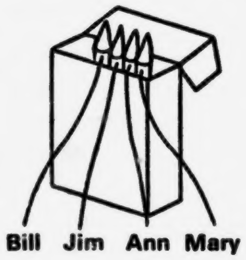
The ideas indicated here form a firm and necessary foundation on which children can build mathematical understanding. Teachers need to structure experiences for the children so that these ideas take on deeper meaning.

Let us review the ideas of sets, matching, number, order, and counting. One activity that might be used is suggested below.

Begin with two sets—a set *A* of children sitting around a table and a set *B* of crayons in a box on the table. Each child is asked to take a crayon from the box. There are three possibilities:

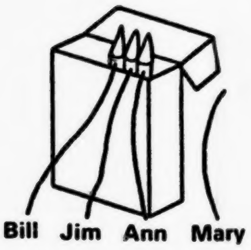


1. There are more crayons than children. The set of children can be matched one-to-one with a proper subset of the set of crayons.



2. There are as many crayons as children. The set of crayons and the set of children are equivalent. (Each member of one set can be paired with a member of the second set in such a manner that each member of the second set is also paired with a member of the first set. In no case are two members of either set paired with the same member of the other set.)

3. There are more children than crayons. The set of crayons can be matched one-to-one with a proper subset of the set of children.



When children attempt to match two given sets *A* and *B*, they soon recognize that there are these three possibilities. There are three corresponding possibilities for any pair of numbers *a* and *b*:

$$a < b, \qquad a = b, \qquad b < a.$$

The teacher should develop situations in which equivalent sets are involved. The children can collect families of equivalent sets. They should be helped to realize that there are equivalent sets in their classroom. For example, there may be four chairs around a table, four books on the table, and so on. The children should be helped to match a set of books with a set of crayons, or with a set of papers, or with some other set. When sets match, they may be thought of as belonging to a particular family of sets. When a child begins to recognize that all these sets in a family have something in common, the child begins to develop an idea of number, such as number 4.

With the idea that some sets can be matched with a proper subset of a set, the idea of ordering of numbers can be developed. For example, children may begin to develop ideas of 4 and 5 from families of sets. Now if they see that set *G* is matched one-to-one with a *proper* subset of *F* (*F* has something "left over"), they can be led to understand that 4 is less than 5, or $4 < 5$.

$$F = \{a,\ b,\ c,\ d,\ e\}.$$
$$G = \{\square, \bigcirc, \triangle, \diamondsuit\}.$$

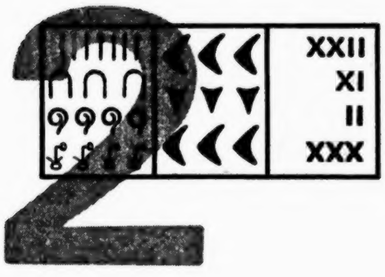
When children begin to understand matching, number, and ordering of numbers, they are ready to count. To count the number of words on this page, they would pair each word with the counting numbers in order: 1, 2, 3, 4, 5, 6,... .

Teachers have realized that THE BIG STEP is taken when children go from pairing objects with objects (for example, chairs with crayons) to pairing members of a set with the counting numbers in order.

The set of numbers $\{1, 2, 3, 4, 5, \ldots\}$ is called the set of counting numbers.

The set consisting of 0 together with all the counting numbers is the set of whole numbers:

$$\{0, 1, 2, 3, 4, \ldots\}.$$



DEVELOPMENT OF OUR DECIMAL NUMERATION SYSTEM

1. **Why is our familiar numeration system called a *decimal* numeration system?**
2. **What is meant by *place value?***
3. **What are the basic ingredients of our decimal numeration system?**
4. **What is *expanded notation?***

Suppose you wanted your pupils to memorize the names of 10,000 persons listed in a telephone book, and to recite these names *in order*? Wild idea? Extremely difficult to do? Yet you may have already taught pupils to recite (if you asked them) many more than 10,000 names in order. These are the names of numbers.

How is it possible for pupils in the years they spend in elementary school to learn to write names for thousands and even millions of numbers? The answer, of course, is that we teach pupils to use a remarkable *system* for representing numbers — the Hindu-Arabic system of numeration with its ten basic symbols or digits.

0, 1, 2, 3, 4, 5, 6, 7, 8, 9

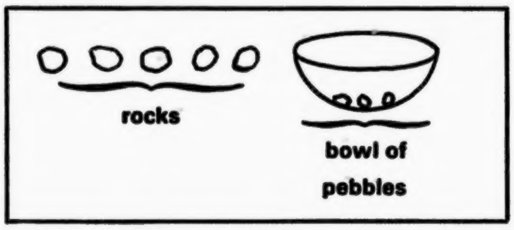
A LOOK BACK

A better appreciation and understanding of our system of numeration may be gained by examining some early systems of numeration. Elements of these early systems play a part in the system we use today.

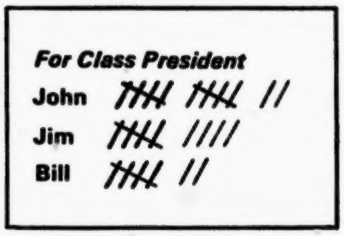
The earliest and simplest system was based upon one-to-one correspondence. For example, when a primitive man saw animals in the forest, he might have wished to let his tribe know how many he saw. He could not carry back the animals to show the tribe, but he could carry back sticks or rocks or any other available objects. He might have discovered he could pair the animals with the fingers of his hands and show how many with his fingers. This would certainly be easier than carrying back rocks.

As the need arose for representing greater numbers, our primitive man ran into trouble. Can you imagine him staggering back to the tribe with 25 rocks, 87 rocks, or perhaps even 100 rocks in his hands to indicate that he had seen that many animals? As man progressed, he had to invent a simple and effective method for representing large numbers.

Imagine how we might check the attendance of a class returning from lunch if we were teaching in those primitive days. We might drop a pebble into a bowl for each pupil entering our "classroom." When the bowl was filled with pebbles, we could place a rock to one side to show the bowl was filled and then empty the bowl. Then we would start over again. In those old days you might have had a class size as shown at the right. The diagram indicates that the number attending was as many as five bowlfuls of pebbles (represented by the five rocks) and three more pebbles in the unfilled bowl. (This crude method assumes, of course, that the bowl held the same number of pebbles each time.)



Manipulative materials alone served man's needs for a long time. Later he invented written symbols as a more convenient way of recording and communicating number ideas. You have probably used one of the earliest and simplest systems for expressing numbers when you have recorded votes for class officers. One symbol (tally mark) is repeated for each vote cast. Now suppose you wished to record John's total vote in a notebook. You could record ////////////, or 𝍸 𝍸 //, as shown in the illustration.



Even ancient civilizations such as the Egyptian, some five thousand years ago, realized that it would be too unwieldy to write as many tally marks as the number of objects recorded. Think of expressing a million in this manner. So the Egyptians decided to use I for one, II for two, III for three, and so on up to nine. However, when they reached ten, they introduced a new symbol. The Egyptians wrote ∩ for ten.

You might wonder why a new symbol was invented for ten and not for four or seven or any other number up to ten. It is generally believed that because the ten fingers were so obviously available to be used for counting, objects were counted and grouped by tens. Therefore a special or new symbol was invented for ten, a different symbol for ten groups of ten, another symbol for ten groups of ten groups of ten, and so on. We can see this dependence upon tens in the ancient Egyptian system of writing number names.

Egyptian System	Hindu-Arabic System
I	1
∩	10
𓍢	100
𓆼	1,000

Now a system requires more than just a set of symbols. It also requires some scheme or plan for combining the symbols. The Egyptians could express all the counting numbers from 1 through 99 with the use of only the two different symbols I and ∩. For example, ∩∩ III meant 23. If ∩ meant ten and I meant one, what plan did the Egyptians use when they wrote ∩∩ III for 23? Since ∩ meant ten, ∩∩ meant ten + ten. Furthermore, III meant one + one + one, or three. So ∩∩III meant 10 + 10 + 1 + 1 + 1, or 23. This is an illustration of what we mean when we say that the ancient Egyptians used an ***addition principle*** in their system of writing numerals.

Exercise Set 1

1. If in the Egyptian system I = 1, ∩ = 10, and 𓍢 = 100, what do the following numerals mean?

a. IIII
b. ∩I
c. ∩∩∩III
d. 𓍢I
e. 𓍢𓍢
f. 𓍢𓍢∩∩II
g. 𓍢𓍢∩∩∩∩IIII
h. 𓍢𓍢𓍢𓍢𓍢𓍢𓍢IIIIIIII
i. 𓍢𓍢𓍢𓍢𓍢∩∩∩∩∩IIIII

2. Express 345 with Egyptian symbols.

3. The Egyptians invented symbols for 1 (I), for 10 (∩), for 10 × 10 (𓍢), for 10 × 10 × 10 (𓆼), for 10 × 10 × 10 × 10 (𓂭), and so on.

a. How did they represent 1,000?

b. How did they represent 10,000?

c. If they were to continue in the same pattern, what did they have to do to represent 100,000?

d. What did they have to do to represent 1,000,000?

e. How many symbols did they need to represent numbers in the millions?

f. What would they have to do to represent numbers in the billions?

4. In the Roman system, I = 1, V = 5, L = 50, C = 100, D = 500, M = 1,000. What do the following numerals mean?

a. II
b. VII
c. XXVII
d. LXVI
e. CCLXXVIII
f. DXII
g. MXII
h. MMCCXI
i. CCCLXXXIII

(Note that each item in this exercise uses an *addition* principle. The Romans also used a *subtraction* principle — for example, in IV, XL, etc.)

5. When we write 23, we mean 20 + 3. In what respect does this resemble the plan used by the Egyptians and the Romans?

6. The Babylonians used ▼ for 1, and ❮ for 10. What number do you think they meant by ❮❮▼▼▼ ?

7. What does the Babylonian numeral ❮❮❮❮▼▼▼▼▼ mean?

8. As shown in Exercises 6 and 7, what plan did the Babylonians use in writing their numerals?

POSITION IN A NUMERAL

If you were a teacher in ancient Egypt, one of your pupils might report that the number of parchments borrowed from Pyramid Library was "∩∩|||." Another pupil might report "|||∩∩." In both reports, the children indicated that 23 parchments were borrowed. Notice that ∩∩||| and |||∩∩ represented the same number, but the numerals looked different. The position of the symbols in the Egyptian system was not important. In our system, however, 23 and 32 are names for different numbers. In our system, the position of symbols is important.

The idea of positional value of number symbols seems to have been used by such ancients as the Babylonians some thousands of years ago. Remember, the Babylonians used ▼ for 1 and ❮ for 10. They repeated symbols and used an additive principle. For numbers 60 or more, they introduced the idea of positional value, as shown in this table.

POSITIONS		NUMBER
Sixty	One	
	❮❮❮▼▼	32
	▼	1
▼		60
	▼▼	2
▼▼		120
▼	▼	61

It is interesting to note that ▼ might represent one or sixty; ▼▼ might represent two, or one hundred twenty, or sixty-one. The *context* enabled one to decide how to interpret the symbol.

Exercise Set 2

1. What number is represented by each of the Babylonian numerals below?

	Sixty	One
a.		❮▼▼
b.	▼	▼▼
c.	❮	❮
d.	❮❮	▼▼▼
e.	❮❮▼	❮❮▼

2. What number is represented by the numeral in the following sentences?

   a. John and Bill are ▼▼ boys.

   b. There are ❮▼▼ eggs in a dozen.

c. There are ▼▼❮❮▼▼▼▼ eggs in a gross.

d. There are ▼ minutes in an hour.

GROUPING, SHORTHAND, AND NUMERALS

We have seen that the ancient Egyptian, Babylonian, and Roman systems for expressing numbers possessed two or more of these features:

1. A small set of number symbols (for example, the Roman symbols I, V, X, L, C, D, M)
2. The addition principle—as used, for example, by the Egyptians (𓍢𓍢∩∩∩| = 100 + 100 + 10 + 10 + 10 + 1.)
3. Positional value—as used, for example, by the Babylonians (▼▼❮❮ = 2 × 60 + 2 × 10.)

Why aren't these systems used today? At the right are some addition examples as we might set them up but expressed with number symbols of the past. Compute the sum in each case. There are no addition facts to learn here. There is nothing to memorize. In each case all a person has to do is to copy down all the symbols. For example, the sum is ∩∩∩|||, or ❮❮❮▼▼▼, or XXXIII. What could be easier?

Egyptian	*Babylonian*	*Roman*
∩∩|| +∩|	❮❮▼▼ +❮▼	XXII + XI

Suppose we want to compute the product of two numbers expressed in Egyptian symbols. We might start the computation as shown at the right. Only the first four partial products are shown. We still have to compute products involving the tens. Care to try to finish the example? It does become somewhat cumbersome to work with these symbols. We can see why the ancients used devices such as the abacus to help them do their computation.

```
  ∩∩∩||||||||
×    ∩∩||||
―――――――――――――
  ∩∩∩||||||||
  ∩∩∩||||||||
  ∩∩∩||||||||
  ∩∩∩||||||||
     . . .
```

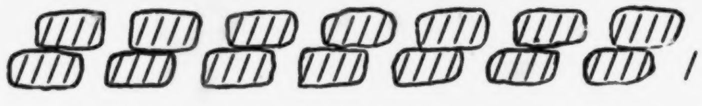
(not completed)

Although it is decidedly more convenient to compute in our (Hindu-Arabic) system than in any of the earlier ones, nevertheless our system actually uses the same fundamental ideas. To see how this comes about, imagine you are back teaching in ancient times. A child takes attendance for you by making a tally mark for each child who enters your "classroom." Here are how many children are present:

///

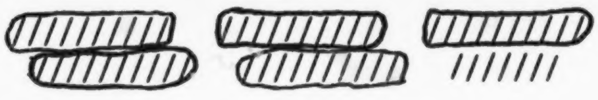
(They had no problem of class size in those days!)

When you ask the child how many are present, he finds it is difficult and hard going to count one by one. So he groups the tally marks



and reports to you: "Fourteen groups of four and one more."

You tell him that the principal has asked all the teachers to report attendance by groups of ten. So he groups the tally marks again



and reports to you: "Five tens and seven more."

You write attendance: . . . . 5 tens + 7 ones

Now imagine that after some time
you get tired of writing 5 tens + 7 ones
and you begin a shorthand: 5 7

On one occasion when you are quite rushed you write: . . . 57

When the principal sees your attendance report, he immediately sends for you.

"What in the world does '57' mean?"

So you explain to the principal your shorthand or abbreviated form for expressing numbers. The *position* of the "5" in "57" is important. The "5" tells the number of tens, and the "7" tells the number of ones. Once the principal understands your abbreviation "57," he asks all teachers to learn the shorthand based on position value of digits. Everyone agrees, once he has learned the plan behind the shorthand, that it saves time to write "57" to mean 5 tens and 7 ones.

Exercise Set 3

1. Group the X's and express the numbers of X's in terms of fours and ones.

X X X X X X X X X X X X X X X X X X X X X X X X X X X X

2. Group the X's by five and express the number of X's in terms of fives and ones.

X X X X X X X X X X X X X X X X X X X X X X X X X X X X

3. Group the X's by tens and express the number of X's in terms of tens and ones.

X X X X X X X X X X X X X X X X X X X X X X X X X X X X

4. Write an abbreviation for 3 tens + 3 ones.

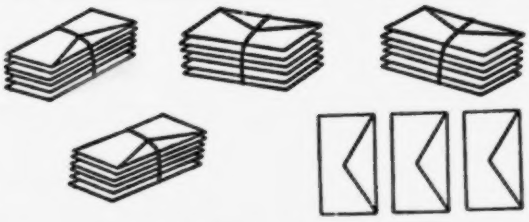
////////// ////////// ////////// ///

5. Write an abbreviation for 4 tens and 2 ones.

////////// ////////// ////////// ////////// //

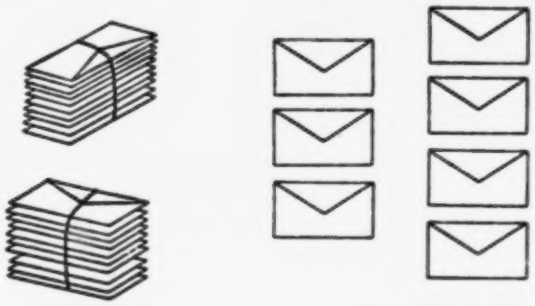
PLACE VALUE

If a stationery sales clerk recorded the number of envelopes sold, he could use any of the ways shown at the right. However, we would know the meaning of the last two abbreviations (read: four-three) only if we knew the clerk's plan for abbreviation. We would have to know that the "4" refers to four sixes.



4 sixes + 3 ones
4 3
43

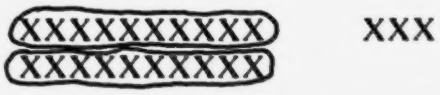
The clerk could also record the same number of envelopes sold (using groups of ten) in any of the ways shown. Again, we would know the meaning of the last two abbreviations only if we knew the clerk's plan for his abbreviation. We would have to know that the "2" refers to 2 tens.



2 packs of ten and 7 more
2 tens + 7 ones
2 7
27

The key to our plan for expressing numbers is the idea of place value, which is based on the way we think of objects' being grouped.

Grouping by tens, it is believed, followed naturally from our having ten fingers. Elements are grouped in sets of ten for ease in counting. Thus, to count eleven we think of one group of ten and one more. Below you see how ten is used to find the number of X's. There are two groups of ten and three more.



We use ten as a "stopping" place in counting. We count in order from one through ten. Then we pause. We continue counting ten and one, ten and two, ten and three, up to ten and nine, and finally two tens. We pause at two tens and then continue: two tens and one, two tens and two, and so on. The idea of grouping by tens, or a "base" ten, is used to set up a place-value system.

The key to our system for naming all whole numbers by using only ten digits (0, 1, 2, 3, 4, 5, 6, 7, 8, 9) is the idea of place value. This idea permits us to abbreviate a numeral

such as 4 tens + 3 ones
to 4 3
or 43

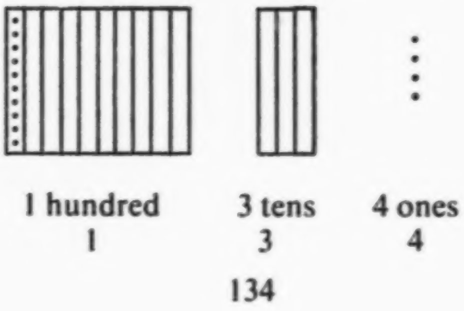
What do we mean by place value? The shorthand "43" means the same thing to all of us only if we know the plan behind the shorthand. Our plan is to assign a number to the position occupied by the "4" and to the position occupied by the "3."

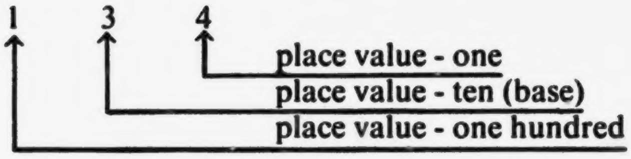
4 3
place value is ten ↑ ↑ place value is one

The number assigned to the position occupied by each digit is the place value of that position.

The "3" occupies the ones place in "43." The "4" occupies the tens place.

Of course, for a large number of objects, groups of ten tens are formed, and so on.




1
3
4
place value - one
place value - ten (base)
place value - one hundred

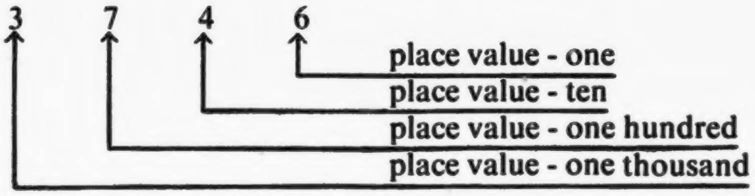
Thus 134 means

$$1 \times 100 \quad + \quad 3 \times 10 \quad + \quad 4 \times 1 \qquad \text{or} \qquad 100 + 30 + 4$$

Each digit in the numeral 134 represents a *product*. In the numeral 134 the digit 1 represents the product 1 × 100; the digit 3 represents the product 3 × 10; the digit 4 represents the product 4 × 1.

To summarize what we have said about place value:

1. Each digit in a numeral occupies a position in the numeral. The *number* assigned to each position is called the *place value* of that position.
2. Each digit in a numeral actually represents a product. It is the product of the number named by the digit and the place value assigned to the position occupied by the digit.


3
7
4
6
place value - one
place value - ten
place value - one hundred
place value - one thousand

Exercise Set 4

1. Write an abbreviation for each of the following:
   - a. 3 tens + 6 ones
   - b. 5 hundreds + 6 tens + 7 ones
   - c. 8 thousands + 6 hundreds + 4 tens + 5 ones
   - d. 4 thousands + 0 hundreds + 7 tens + 3 ones
   - e. $3 \times 10 \quad + \quad 6 \times 1$
   - f. $5 \times 100 \quad + \quad 6 \times 10 \quad + \quad 3 \times 1$
   - g. $8 \times 1{,}000 \quad + \quad 7 \times 100 \quad + \quad 6 \times 10 \quad + \quad 4 \times 1$
   - h. $5 \times 1{,}000 \quad + \quad 6 \times 10 \quad + \quad 8 \times 1$

2. What product is represented by each digit in the numeral 347?

3. What product is represented by each digit in the numeral 4,916?

4. What product is represented by each digit in the numeral 3,033?

PLACE-VALUE CHART

To help children understand our plan for writing numerals, we can display our plan in the form of a place-value chart. Such a chart indicates the number to be assigned to each position occupied by a digit in a numeral such as 56,342. The key to understanding a place-value chart is to understand the part played by our base number — ten.

PLACE-VALUE CHART

Ten × ten × ten × ten	Ten × ten × ten	Ten × ten	Base ten	One
10,000	1,000	100	10	1
5	6	3	4	2

What pattern do you see in our plan for assigning numbers to each position of a digit in a numeral? As we move toward the left from the "one" position, each new position is assigned a value ten times as great as the number assigned to the previous position.

Our system of writing numerals is called a decimal system. It is based on tens. Systems based on numbers other than ten could be, and have been, devised.

Exercise Set 5

1. What product is represented by each digit in

a. 56,342? **b.** 20,518?

2. How would you find the value of the position immediately to the left of the ten x ten x ten x ten place?

3. Examine the place-value charts below. Each is built upon a grouping by other than ten. In each chart, what is the value of the next three positions immediately to the left of the base place?

a.

PLACE-VALUE CHART

Eight × eight × eight	Eight × eight	Base eight	One

b.

PLACE-VALUE CHART

Seven × seven	Base seven	One

c.

PLACE-VALUE CHART

	Base five	One

d.

PLACE-VALUE CHART

	Base six	One

STANDARD FORM AND EXPANDED FORM

We have all used such standard abbreviations as "in." for inch, "yd." for yard, and "ft." for foot. It is convenient to use abbreviations provided there is general agreement as to the meaning of the abbreviation.

A numeral such as 34 is the standard numeral for the number thirty-four. The numeral 34 is an abbreviation (standard) for "30 + 4," or "(3 × 10) + (4 × 1)." We use the standard numeral 34 for convenience and for uniformity of response.

Two different numerals for the number thirty-four are "30 + 4" and "34." The usual way of asserting that these two numerals name the same number is to write 34 = 30 + 4.

"34" is called a standard numeral.

"30 + 4" is called an expanded form of the standard numeral 34.

To express a standard numeral in expanded form, it is necessary to think of the place value of each digit. For example,

$$347 = 3 \times 100 \quad + \quad 4 \times 10 \quad + \quad 7 \times 1 \qquad \text{or} \qquad 300 + 40 + 7.$$

In 347, the "3" represents the product 3 × 100; the "4" represents the product 4 × 10; the "7" represents 7 × 1.

Notice that the value of the numeral is the *sum* of the products represented by each of its digits, according to the position of each digit in the numeral.

We saw previously that the idea of adding the values of each symbol was used in earlier systems also.

∩∩ III = 10 + 10 + 1 + 1 + 1.
XXXII = 10 + 10 + 10 + 1 + 1.
57 = 50 + 7 or
5 x 10 + 7 x 1.

The form of the expanded numeral will vary depending upon the grade level and previous learning of the children as well as the preference of the teacher. Let us illustrate with the standard numeral 356.

PLACE-VALUE CHART

Ten x ten	Base ten	One
3	5	6

$$356 = 3 \times 100 + 5 \times 10 + 6 \times 1$$
$$= 300 + 50 + 6.$$

From our place-value chart, the standard numeral "356" means

$$356 = 3 \times (10 \times 10) + 5 \times 10 + 6 \times 1.$$

Thus, we have several expanded forms for the numeral 356.

$$\begin{aligned} 356 &= 3 \times (10 \times 10) + 5 \times 10 + 6 \times 1 \\ &= 3 \times 100 + 5 \times 10 + 6 \times 1 \\ &= 300 + 50 + 6. \end{aligned}$$

Consider another illustration.

PLACE-VALUE CHART

Ten x ten x ten	Ten x ten	Base ten	One
4	5	3	6

STANDARD NUMERAL — EXPANDED FORMS

$$\begin{aligned} 4{,}536 &= 4{,}000 + 500 + 30 + 6 \\ &= 4 \times 1{,}000 + 5 \times 100 + 3 \times 10 + 6 \times 1 \\ &= 4 \times (10 \times 10 \times 10) + 5 \times (10 \times 10) + 3 \times 10 + 6 \times 1 \end{aligned}$$

In the last expanded form, notice the "10 x 10 x 10" and "10 x 10." We can abbreviate 10 x 10 x 10 as 10^3 (read "ten cubed") and 10 x 10 as 10^2 (read "10 squared").

In 10^3 and 10^2 the 3 and the 2 are called exponents. If we use exponents, our place-value table begins to look like this:

PLACE-VALUE CHART

Ten x ten x ten	Ten x ten	Base ten	One
10^3	10^2	10	1
5	4	0	3

$$\begin{aligned} 5{,}403 &= 5{,}000 + 400 + 0 + 3 \\ &= 5 \times 1{,}000 \quad + \quad 4 \times 100 \quad + \quad 0 \times 10 \quad + \quad 3 \times 1 \\ &= 5 \times (10 \times 10 \times 10) \quad + \quad 4 \times (10 \times 10) \quad + \quad 0 \times 10 \quad + \quad 3 \times 1 \\ &= 5 \times 10^3 \quad + \quad 4 \times 10^2 \quad + \quad 0 \times 10 \quad + \quad 3 \times 1. \end{aligned}$$

The use of exponents helps us to simplify our expanded numerals. Examine the place-value chart below.

PLACE-VALUE CHART

Ten x ten x ten x ten	Ten x ten x ten	Ten x ten	Base ten	One
10^4	10^3	10^2	10	1
3	4	2	8	9

Standard numeral 34,289

$$\begin{aligned} &= 30{,}000 &&+ 4{,}000 &&+ 200 &&+ 80 &&+ 9 \\ &= 3 \times 10{,}000 &&+ 4 \times 1{,}000 &&+ 2 \times 100 &&+ 8 \times 10 &&+ 9 \times 1 \\ &= 3 \times (10 \times 10 \times 10 \times 10) &&+ 4 \times (10 \times 10 \times 10) &&+ 2 \times (10 \times 10) &&+ 8 \times 10 &&+ 9 \times 1 \\ &= 3 \times 10^4 &&+ 4 \times 10^3 &&+ 2 \times 10^2 &&+ 8 \times 10 &&+ 9 \times 1 \end{aligned}$$

In the last expanded numeral for "34, 289" you will notice how the exponents decrease in order from left to right. In fact, this continues past the 2×10^2 term. We *define* 10^1 as 10 and 10^0 as 1. Therefore 8×10 may be written 8×10^1, and 9×1 may be written 9×10^0.

STANDARD NUMERAL	EXPANDED NUMERAL
34,289 =	$3 \times 10^4 + 4 \times 10^3 + 2 \times 10^2 + 8 \times 10^1 + 9 \times 10^0$.

As we said earlier, the form of the expanded numeral used in your class will depend upon grade level, previous learning, and teacher preference. The various forms have been shown in order to suggest how the child's understanding of our numeration system develops.

We have seen that a number can be named in many ways. For example, it can be named by a standard numeral (57) or by the expanded numeral (50 + 7). To show that two numerals name the same number, we use an equality sign (=) and write 57 = 50 + 7. If we write the sentence 642 = 600 + 40 + 2, we are stating that the numeral 642 and the numeral 600 + 40 + 2 name the same number.

Some mathematical sentences, such as $57 = 50 + 7$, $642 = 600 + 40 + 2$, $6 = 5 + 1$, and $8 = 5 + 4$, can be judged *true* or *false*. For example, judge these statements true or false:

1. $734 = 700 + 30 + 4$. (True)
2. $652 = 6 \times 10^2 + 5 \times 10 + 2 \times 1$. (True)
3. $8{,}316 = 8 \times 10^2 + 3 \times 10 + 16$. (False)

We often encounter mathematical sentences, such as $6 + \square = 10$, that show a relationship in a problem. (If a boy has 6¢, how much more does he need in order to buy a 10¢ notebook?) The symbol $\square$ is usually called a frame. The sentence $6 + \square = 10$ cannot be judged true or false until a numeral is placed in the frame. We say the truth or falsity of the sentence is *open* until we "fill in" the frame. Sentences such as $6 + \square = 10$ are called *open sentences*. If we fill in the frame with a "5," we have $6 + \boxed{5} = 10$, which is a false statement. If we fill in the frame with a "4," we have $6 + \boxed{4} = 10$, which is a true statement.

Exercise Set 6

1. Write a place-value chart going from ones through to ten thousands using a base ten. Use both the notation of ten × ten × ten and also the exponent notation, such as 10^3, and so on.

2. Write four expanded numerals for each of the following:

a. 146 **b.** 329 **c.** 7,146 **d.** 33,412 **e.** 296,314

3. Write standard numerals for the following:

a. $4 \times 100 + 3 \times 10 + 6 \times 1$
b. $6 \times 1{,}000 + 5 \times 100 + 4 \times 10 + 7 \times 1$
c. $7{,}000 + 600 + 50 + 8$
d. $5{,}000 + 60 + 9$
e. $6 \times 10{,}000 + 5 \times 1{,}000 + 4 \times 100 + 3 \times 10 + 6 \times 1$
f. $4 \times 1{,}000 + 6 \times 100 + 7 \times 10 + 6 \times 1$
g. $5 \times 10{,}000 + 0 \times 1{,}000 + 0 \times 100 + 6 \times 10 + 0 \times 1$
h. $4 \times 10^3 + 3 \times 10^2 + 6 \times 10^1 + 1$
i. $7 \times 10^2 + 4 \times 10^1 + 7 \times 10^0$ (Remember $10^0 = 1$)
j. $8 \times 10^3 + 5 \times 10^2 + 6 \times 10^1 + 7 \times 10^0$
k. $9 \times 10^5 + 6 \times 10^4 + 3 \times 10^3 + 2 \times 10^2 + 5 \times 10^1 + 6 \times 10^0$

l. $7 \times 10^6 + 3 \times 10^5 + 6 \times 10^4 + 9 \times 10^3 + 6 \times 10^2 + 0 \times 10^1 + 0 \times 10^0$

m. $8 \times 10^6 + 0 \times 10^5 + 0 \times 10^4 + 3 \times 10^3 + 0 \times 10^2 + 5 \times 10^1 + 6 \times 10^0$

4. Fill in the frame in each open sentence so that a true statement results.

a. $24 = (2 \times \square) + 4.$

b. $93 = (\square \times 10) + 3.$

c. $146 = (\square \times 10^2) + (4 \times 10) + (6 \times 1).$

d. $347 = (3 \times \square) + (4 \times 10) + (7 \times 1).$

e. $4{,}569 = (4 \times 10^3) + (5 \times \square^2) + (6 \times 10) + (9 \times 1).$

f. $3{,}981 = (3 \times \square) + (9 \times 10^2) + (8 \times 10^1) + (1 \times 10^0).$

If two □'s appear in the same open sentence, each □ is to be filled by a name for the same number.

g. $6{,}343 = (6 \times 1{,}000) + (\square \times 100) + (4 \times 10) + (\square \times 1).$

h. $54{,}649 = (5 \times 10^4) + (\square \times 10^3) + (6 \times 10^2) + (\square \times 10^1) + (9 \times 10^0).$

i. $34{,}162 = (3 \times \square^4) + (4 \times \square^3) + (1 \times \square^2) + (6 \times \square^1) + (2 \times \square^0).$

5. Examine a pupil's work below:

$$\begin{array}{r} 36 \\ \underline{+\,27} \\ 513 \end{array}$$

Show how the use of an expanded form could help the child arrive at a correct standard numeral for the answer.

SUMMARY

We have examined our (Hindu-Arabic) system of numeration. A child who understands our decimal numeration system can represent any whole number quickly and conveniently. The system is built upon a few basic ideas:

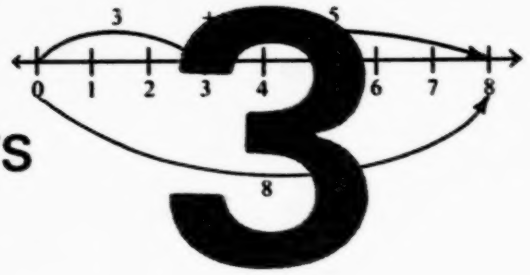
1. Base ten—group or count by tens: ten × ten, ten × ten × ten, etc.
2. A set of ten digits—0, 1, 2, 3, 4, 5, 6, 7, 8, 9

3. Place value
   - *a*) A number (place value) is assigned to each position in the numeral.
   - *b*) Each digit represents a *product* — the product of its value and the place value assigned to the position occupied by the digit.

4. The idea that the value of a numeral is the *sum* of the products represented by the digits of the numeral

A system such as ours, which is based upon groups of ten, is called a decimal system of numeration.

Thus, with the aid of ten symbols, the idea of place value, and the use of addition and multiplication, all whole numbers can be represented. We have seen that ancient systems used some of the basic ideas listed above. It is possible, and in fact extremely useful for computers, to develop systems of numeration that do not depend upon base ten. Computers usually use base two because their circuits are often built up from "two-state" switching devices. You might find it interesting and challenging to modify (1), (2), and (3) above and develop a system of numeration based on groups of five (fingers on one hand), five x five, and so on.



3

ADDITION AND ITS PROPERTIES

1. **What is meant by the union of a pair of sets?**
2. **How can the sum of whole numbers be defined in terms of sets?**
3. **What are some properties of addition?**
4. **What does an expression such as "2 + 3" mean?**

$$\begin{array}{r} 623 \\ +\,108 \\ \hline 731 \end{array} \qquad \begin{array}{r} 821 \\ -\,538 \\ \hline 283 \end{array}$$

$$\begin{array}{r} 362 \\ \times\ \ 27 \\ \hline 2{,}534 \\ 724 \\ \hline 9{,}774 \end{array} \qquad \begin{array}{r} 18 \\ 23\overline{)414} \\ 23 \\ \hline 184 \\ 184 \\ \hline \end{array}$$

$$\begin{array}{r} 12 \\ +\,15 \\ \hline 27 \end{array}$$

Do you know a child who is able to do the work shown at the right but who has trouble with verbal problems? Imagine that you have a pupil who has done the work shown. Then imagine that you give him the following problem: How many seats are there in an auditorium in which there are 15 rows of seats and there are 12 seats in each row? Now suppose he writes on his paper . . . . and says that the answer is 27. How do you analyze his difficulties? Is he careless, or does he lack an understanding of addition and multiplication? Isn't it possible for a child to do the work shown at the upper right without really understanding the meaning of the operations in mathematics?

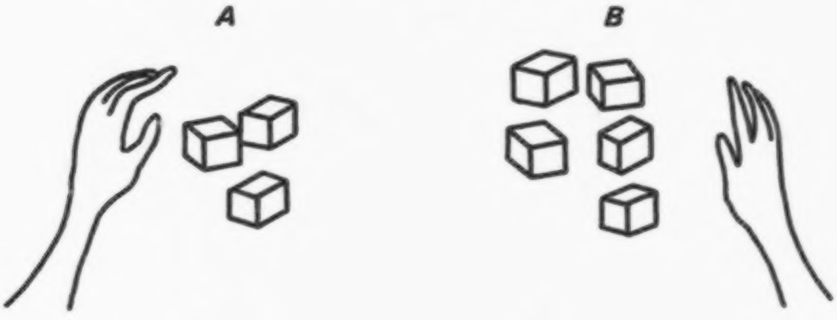
You will, of course, agree that all the computational skills in the world are not of much help if they aren't accompanied by understanding of the results. If a child knows *how* to multiply but not *when* to multiply, his knowledge is rather useless.

The point we are driving at is that *there is a difference between knowing the meaning of addition and knowing how to carry out the related computational process.* Although it is possible to learn the latter by rote, it is doubtful if any worthwhile educational objectives are attained in doing so.

In the following pages we shall try to make clear what we mean by addition. Addition will be developed through the use of sets. We shall see that we can then state certain principles or properties of addition as a consequence of this development. These properties ultimately lead us to the processes that we use in computation.

PRELIMINARIES TO ADDITION—UNION OF SETS

What do we mean when we speak of "adding 3 to 5"? Usually a teacher in the primary grades explains by showing a set of 3 objects and a set of 5 other objects. Upon joining the two sets, a new set of 8 objects is produced. Using physical examples of this kind is a good classroom technique, but as a teacher you will want to understand the mathematical ideas that underlie such physical examples.



Let's examine this situation and see if we can state precisely what addition is all about. Suppose we abide by convention and name each of the original sets above with a capital letter. We might arbitrarily call the set shown on the left in the picture, A, and the one on the right, B. Then the set consisting of all the elements shown is called the *union* of A and B and denoted $A \cup B$ (read "A union B").

Let's consider several more examples that illustrate the concept of union:

1. If E is the set of all blonds in the class and F is the set of all redheads, then $E \cup F$ is the set of all those in the class who have either blond or red hair.

2. Suppose M is the set consisting of Bob and Joe and N is the set consisting of Betty, Jean, and Mary. Using braces in the customary manner, we might write $M = \{\text{Bob, Joe}\}$ and $N = \{\text{Betty, Jean, Mary}\}$. Then $M \cup N = \{\text{Bob, Joe, Betty, Jean, Mary}\}$.

3. Let X be the set of states in the United States whose names begin with "C"; that is, $X = \{\text{California, Colorado, Connecticut}\}$. Let Y be

the set of states whose names end with "t"; that is, Y = {Connecticut, Vermont}. Then $X \cup Y$ = {California, Colorado, Connecticut, Vermont}.

4. If $P = \{a, b, c\}$ and $Q = \{a, c, e, g\}$, then $P \cup Q = \{a, b, c, e, g\}$.

You probably recognized that the last two examples differ from the first two. In example 3, set X contains Connecticut and so does Y. But Connecticut is not listed more than once in tabulating the elements in the union. In 4, sets P and Q have two elements in common, namely, a and c. Again these are listed but once in $P \cup Q$. We are thus saying that the union of any set A and any set B is the set consisting of all the elements in A together with all the elements in B. Among the elements included in the union are, of course, any elements that happen to be common to A and B. However, an element of the union is listed but once, regardless of whether it belongs to only one of the sets or to both. This idea is briefly expressed as follows: "The union of sets A and B is the set consisting of all those elements that are in A *or* in B." (As used here, the word "*or*" does not exclude the possibility that an element of the union might belong to both sets.) Notice that a *physical act* of joining is *not* implicit in the concept of union of two sets. This is certainly the case in example 3 above.

Exercise Set 1

Let $A = \{a, b, c\}$, $B = \{a, e, i, o, u\}$, $C = \{b, f, g\}$, and $D = \{u, v, w, x, y, z\}$. Tabulate each of the following sets:

1. $A \cup B$
2. $B \cup A$
3. $A \cup C$
4. $B \cup C$
5. $A \cup D$
6. $(A \cup B) \cup C$ First determine the union of A and B, then the union of that set and C.
7. $A \cup (B \cup C)$
8. $B \cup \{\ \}$ Note: $\{\ \}$ is the empty set, the set that has no members.

ADDITION

Returning now to the explanation of "adding 3 and 5," we can say that the teacher selects a set A with 3 elements and a set B with 5 *other* elements. We can write $n(A) = 3$ (read "the number of elements in A is 3") and $n(B) = 5$. The set consisting of all the elements in A as well as those in B is $A \cup B$. Then $n(A \cup B)$—the number of elements in $A \cup B$—is what we call the *sum* of 3 and 5. This sum is denoted by "$3 + 5$" (read "3 plus 5"). And since by counting we find that $n(A \cup B)$ is 8, we write

$$3 + 5 = 8.$$

In general, we would like to be able to say something like this: For any two numbers a and b, we choose a set A that contains a elements—that is, $n(A) = a$—and a set B with b elements—that is, $n(B) = b$. Then $a + b = n(A \cup B)$. However, such a definition presents one difficulty. Suppose we wish to determine the sum of 3 and 5 and for our sets we select A and B from the exercises above. Since $A = \{a, b, c\}$ and $B = \{a, e, i, o, u\}$, clearly $n(A) = 3$ and $n(B) = 5$. Then $A \cup B = \{a, b, c, e, i, o, u\}$. So $n(A \cup B) = 7$. But we would probably react unfavorably to

$$3 + 5 = 7.$$

Of course, you recognize that the difficulty stems from the fact that in the latter example we selected sets that have an element in common. When a teacher uses actual physical objects to demonstrate in class, this problem usually does not arise; but it must be considered in stating a definition. When two sets have no common elements, they are said to be *disjoint* or *mutually exclusive*. The definition of a sum of 3 and 5 could then be stated as follows: Let A and B be disjoint sets such that $n(A) = 3$ and $n(B) = 5$. Then $3 + 5 = n(A \cup B)$.

The definition of the sum of any pair of whole numbers is as follows:

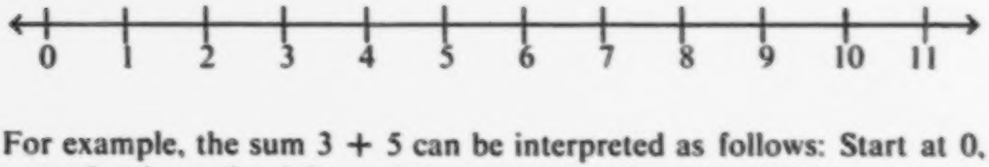
> *If a and b are any whole numbers, let A and B be disjoint sets such that $n(A) = a$ and $n(B) = b$. The sum of a and b, denoted "$a + b$," is $n(A \cup B)$.*

Notice the important role played by sets in defining a sum.

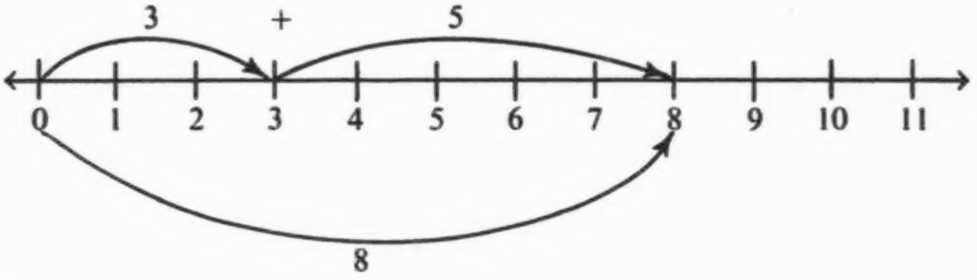
The sum $a + b$ does not depend on the nature of the a elements which comprise set A nor on the nature of the b elements which comprise set B, so long as these two sets are disjoint.

The assignment of a sum to a pair of numbers is essentially what we mean by *addition*.

Another approach to addition, one that is becoming popular in modern mathematics programs, makes use of the "number line."



For example, the sum $3 + 5$ can be interpreted as follows: Start at 0, move 3 units to the right, and then move 5 units to the right:



The result is the same as that of a single movement of 8 units to the right. The advantage of starting at 0 is that the computed result 8 is immediately obtained by inspection of the number line.

Exercise Set 2

1. Tell what $7 + 2$ means in terms of sets.

2. It is important to distinguish between the language and symbols that apply to sets and the language and symbols that apply to numbers. If capital letters represent sets, which of the following are meaningless according to the definitions we have given?

- **a.** The union of *M* and *N*
- **b.** The union of 6 and 5
- **c.** $3 \cup 4$
- **d.** $7 + 6$
- **e.** $n(E) \cup n(F)$
- **f.** $P + Q$
- **g.** $n(P) + n(Q)$
- **h.** The sum of 8 and 3
- **i.** The sum of *R* and *S*
- **j.** $E \cup F$

3. Suppose A is a set such that $n(A) = 5$ and B is a set such that $n(B) = 7$. If $n(A \cup B) = 10$, what can you say about sets A and B?

4. Is it possible to find two sets A and B for which $n(A) + n(B) < n(A \cup B)$? Explain.

5. Use a number line to depict the following sums:
   a. $2 + 4$ b. $4 + 2$ c. $5 + 1$ d. $3 + 3$

THE COMMUTATIVE PROPERTY OF ADDITION

The development of addition through the use of sets makes it possible to derive some of the characteristic properties of addition. The first property to be discussed is exemplified by the statement

$$7 + 2 = 2 + 7.$$

This statement illustrates the *commutative property* of addition. Although the fact that $7 + 2 = 2 + 7$ is obvious to anyone familiar with addition, it is not so obvious to the beginner. In fact, most first-grade children will readily determine that $7 + 2 = 9$ but will quite often have trouble with $2 + 7$. Consequently, the commutative property should be emphasized early in arithmetic. In general terms, the commutative property of addition is stated as follows:

If a and b are whole numbers, then $a + b = b + a$.

Because of this property, we say: "Addition is commutative."

We can justify the commutative property by making use of the definition of sum. Let us refer back to Exercises 1 and 2 in Set 1. You were given that $A = \{a, b, c\}$ and $B = \{a, e, i, o, u\}$. So you might have written

$$A \cup B = \{a, b, c, e, i, o, u\}.$$
$$B \cup A = \{a, e, i, o, u, b, c\}.$$

But no matter how you wrote things down, $A \cup B$ and $B \cup A$ contain exactly the same elements. We thus write

$$A \cup B = B \cup A.$$

It isn't hard to see that this is true for any sets A and B. The set of all elements to be found in either A or B (including, of course, any elements that may be common to both) is the same as the set of all elements in B or in A (or in both).

From the fact that, for any sets A and B, $A \cup B = B \cup A$, the commutative property of addition follows. To verify, for example, that

$$7 + 2 = 2 + 7,$$

we select a set A with 7 elements and a set B disjoint from A with 2 elements. Then $7 + 2 = n(A \cup B)$ while $2 + 7 = n(B \cup A)$. But since $A \cup B = B \cup A$, it follows that $n(A \cup B) = n(B \cup A)$ and therefore $7 + 2 = 2 + 7$. In general terms, if a and b are any two whole numbers, we select disjoint sets A and B such that $n(A) = a$ and $n(B) = b$. Then $a + b = n(A \cup B)$ and $b + a = n(B \cup A)$. Again we have

$$A \cup B = B \cup A,$$

from which it follows that

$$n(A \cup B) = n(B \cup A),$$

and thus

$$a + b = b + a.$$

With children in the elementary grades, this property might be expressed with frames:

$$\square + \triangle = \triangle + \square$$

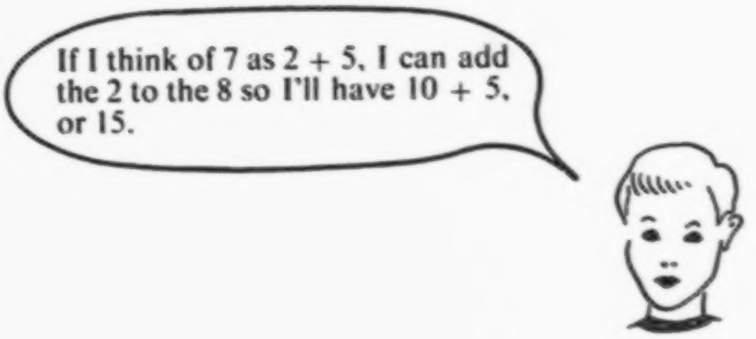
When working with such frames, it is agreed that the same number is to be used for a frame of a particular shape wherever that shape appears in a given sentence.

The importance of this property becomes more and more evident as a child advances in school. However, the primary teacher should realize that the simple fact that it greatly reduces the memorization task faced by the pupil is reason enough for stressing the commutative property early.

THE ASSOCIATIVE PROPERTY OF ADDITION

There are other important properties of addition, and these also can be used by a child early in his study of arithmetic. Some children, when asked what is the sum of 8 and 7, will think as the boy does in the picture below:

$8 + 7 = \square.$


If I think of 7 as 2 + 5, I can add the 2 to the 8 so I'll have 10 + 5, or 15.

This boy is tacitly making use of a grouping idea that mathematicians call the *associative property of addition*. He is thinking of 7 as $2 + 5$ and then reasoning that

$$8 + (2 + 5) = (8 + 2) + 5.$$

The general statement of the associative property would be:

If a, b, and c are whole numbers,
then $(a + b) + c = a + (b + c)$.

Because of this property, we say: "Addition is associative." Again, the relationship can be justified by using the definition of sum. In Exercise Set 1, examples 6 and 7, you should have found that

$$(A \cup B) \cup C = A \cup (B \cup C).$$

You see that this would be true for any sets A, B, and C, because the expression on either side of the equation represents the set of all elements in A or in B or in C. Now, given any numbers a, b, and c, we can choose sets A, B, and C with no elements in common so that $n(A) = a$, $n(B) = b$, and $n(C) = c$. Then

$$n[(A \cup B) \cup C] = n(A \cup B) + n(C) = (a + b) + c,$$

and

$$n[A \cup (B \cup C)] = n(A) + n(B \cup C) = a + (b + c).$$

Therefore,

$$(a + b) + c = a + (b + c).$$

This too might be expressed with frames:

$$(\square + \triangle) + \diamond = \square + (\triangle + \diamond).$$

Because of the associative property, no ambiguity results if parentheses are omitted from an expression for a sum. For example, since

$$(5 + 3) + 9 = 5 + (3 + 9)$$

we could write

$$5 + 3 + 9$$

to represent either expression. Note, though, that this is not always the case in mathematics. Consider division. Notice that

$$(24 \div 6) \div 2 = 4 \div 2 = 2$$

while

$$24 \div (6 \div 2) = 24 \div 3 = 8.$$

Since

$$(24 \div 6) \div 2 \neq 24 \div (6 \div 2),$$

division is not associative and we don't write $24 \div 6 \div 2$ without some agreement as to grouping.

Children might refer to the commutative and associative properties simply as the "order" and "grouping" properties.

Often the commutative and associative properties can be used together advantageously. For example, in computing $7 + (9 + 3)$, it is easier if 3 is grouped with 7; but this involves reordering and regrouping (whether we actually write it down or not):

$7 + (9 + 3) = 7 + (3 + 9)$	by the commutative property of addition
$= (7 + 3) + 9$	by the associative property of addition
$= 10 + 9$	because $7 + 3 = 10$, and
$= 19$	because $10 + 9 = 19$ by our system of numeration.

When you tell a child to "check by adding up," you are utilizing both properties. Consider, for instance, the units column in the example below:

$$\begin{array}{r}27\\54\\\underline{32}\end{array}$$

Working downward, we must think of the sum $(7 + 4) + 2$. Working upward, we have $(2 + 4) + 7$. We know they are the same because of the commutative and associative properties:

$(7 + 4) + 2 = 2 + (7 + 4)$	addition is commutative ($7 + 4$ is interchanged with 2)
$= 2 + (4 + 7)$	addition is commutative
$= (2 + 4) + 7$	addition is associative.

To compute the sum of 23 and 45, some children think "$23 + 40$ is 63, and $63 + 5$ is 68." Let's analyze this. First,

$25 + 45 = 23 + (40 + 5)$	because $45 = 40 + 5$ by our system of numeration
$= (23 + 40) + 5$	addition is associative.

Now to determine that $23 + 40 = 63$ (or $60 + 3$) we reason:

$23 + 40 = (20 + 3) + 40$	because $23 = 20 + 3$ by our system of numeration
$= 20 + (3 + 40)$	addition is associative
$= 20 + (40 + 3)$	addition is commutative

$= (20 + 40) + 3$ addition is associative

$= 60 + 3$ because $20 + 40 = 60$.

If we then replace $23 + 40$ by $60 + 3$ in the expression $(23 + 40) + 5$, we have

$$\begin{aligned}(23 + 40) + 5 &= (60 + 3) + 5 \\ &= 60 + (3 + 5) && \text{addition is associative} \\ &= 60 + 8 && \text{because } 3 + 5 = 8 \\ &= 68 && \text{because } 60 + 8 = 68.\end{aligned}$$

After some work of this sort, one usually becomes convinced that the commutative and associative properties justify rearranging (in any manner we choose) the terms in an expression for a sum. This can indeed be shown to be the case. Although we don't intend to present a detailed proof, we shall feel free to use the rearranging idea for addition henceforth. For example, we might say that

$$[(7 + 1) + (4 + 9)] + (3 + 6) = (7 + 3) + [(9 + 1) + (6 + 4)]$$

by the commutative and associative properties. Moreover, since it does not matter in a sum how the numbers are grouped, nor how they are ordered, grouping symbols may be omitted and computation may be carried out in any order.

Exercise Set 3

1. Identify the property exemplified by each of the following:
   - a. $7 + 9 = 9 + 7$.
   - b. $(2 + 3) + 8 = 2 + (3 + 8)$.
   - c. $(4 + 7) + 1 = (7 + 4) + 1$.
   - d. $(2 + 9) + (3 + 1) = (3 + 1) + (2 + 9)$.
   - e. $6 + (4 + 9) = (6 + 4) + 9$.
   - f. $6 + (5 + 4) = (5 + 4) + 6$.

2. Show how the associative and/or commutative properties can be used to simplify the computation of these sums:
   - a. $7 + (3 + 6)$
   - b. $8 + (5 + 2)$
   - c. $(4 + 9) + 1$

d. $17 + (28 + 3)$

e. $(16 + 7) + (3 + 4)$

3. Suppose that $a \star b$ means "2 times the sum of a and b." Examples:

$$1 \star 5 = 12.$$
$$4 \star 0 = 8.$$
$$3 \star 7 = 20.$$

a. Is the operation denoted by "$\star$" commutative? How would you justify your answer?

b. To compute $(2 \star 3) \star 4$, we first compute $2 \star 3$. Since this is 10,

$$(2 \star 3) \star 4 = 10 \star 4 = 28.$$

Compute $2 \star (3 \star 4)$.

c. Is the operation denoted by "$\star$" associative? Explain.

THE ADDITION PROPERTY OF 0

In Exercise 8 of Set 1, on page 36, you should have found that $B \cup \{\ \} = B$. Since the empty set has no elements, the union of any set A with the empty set will be A; that is,

$$A \cup \{\ \} = A.$$

This leads to another important property of addition involving the number 0. For any number a, we can select a set A with a elements. Now $\{\ \}$ is a set with no members, and it clearly has no elements in common with any other set. By the definition of sum,

$$a + 0 = n(A \cup \{\ \}).$$

But if

$$A \cup \{\ \} = A,$$

then

$$n(A \cup \{\ \}) = n(A),$$

so

$$a + 0 = a.$$

We shall call this the *addition property of 0*. Because the number 0 behaves in this special way, it is called the *identity element for addition* or the *additive identity*. The identity element is also called the *neutral element*.

It should be clearly understood that 0 is a perfectly good *number*. "Zero" does not mean "nothing"!

Exercise Set 4

1. There are 100 addition "facts"—from $0 + 0 = 0$ up to $9 + 9 = 18$— that children are expected to memorize. If a child learns the commutative property and the addition property of 0, how many essentially different facts are there?

2. Explain the meaning of $a < b$ in terms of addition. (Assume that a and b are whole numbers.)

MULTIPLICATION AND ITS PROPERTIES

1. **How can the meaning of the product of two whole numbers be conveyed**
   - ***a*) through disjoint sets?**
   - ***b*) through cross products of sets?**
   - ***c*) through arrays?**
2. **What are some properties of multiplication?**
3. **What does an expression such as "2 x 3" mean?**

If you, an adult, are asked to multiply 3 and 2, you will quickly think of 6. Probably you will not bother to think of how the 6 is determined; you know it too well to need to think about it. But children don't know anything about multiplication until they learn from adults. How should children be taught that 6, rather than some other number, is the product of 3 and 2?

More generally, the question we need to ask and answer for children is "What does multiplication mean?"

In a previous section we discussed a similar question about addition. Briefly, we recall that—

1. By the *sum* of 3 and 2 (that is, 3 + 2) we mean the number of elements in the *union* of a set of 3 members and another set (disjoint from the first) with 2 members. By actual count we find that the number of elements in the union is 5. Hence, 3 + 2 = 5.

2. With a pair of whole numbers (addends), addition associates a whole number (their sum).

Multiplication can be handled in a similar manner. We shall see that—

1. The meaning of a "product" (such as 3 x 2) can also be revealed through the use of sets.

2. With a pair of whole numbers ("factors"), multiplication associates a number (their "product").

Various approaches to multiplication are possible. Some newer programs employ the *cross product* because of certain advantages it has over more traditional approaches.

CROSS PRODUCT

Consider this situation: A man is going to eat a sandwich. He has a choice of hamburger, salami, or tuna. After eating the sandwich, he will drink either coffee or milk. What are all the combinations of sandwich and beverage he may choose? Remember, he is going to eat the sandwich first, then drink the beverage.

Here are all the possibilities:

(hamburger, coffee)	(salami, coffee)	(tuna, coffee)
(hamburger, milk)	(salami, milk)	(tuna, milk)

We may think of all these possibilities as forming a set. This set has *pairs* for its elements, and the set has six of these pairs. For example, the pair (hamburger, coffee) is a single element of the set.

What is the mathematical significance of the above situation?

We are given two sets: a set of sandwiches,

{hamburger, salami, tuna},

and a set of beverages,

{coffee, milk}.

From these two sets we determine a third set (a set whose elements are pairs made of elements of the two given sets):

{(hamburger, coffee), (hamburger, milk),
(salami, coffee), (salami, milk),
(tuna, coffee), (tuna, milk)}.

In general terms, we may say that from any two sets we can determine in the same way a set of pairs. The set of pairs is called the *cross product* of the given sets. We use the symbol × between the given sets to name the cross product:

{hamburger, salami, tuna} × {coffee, milk} =
{(hamburger, coffee), (hamburger, milk),
(salami, coffee), (salami, milk),
(tuna, coffee), (tuna, milk)}.

Let us abbreviate this rather lengthy sentence by writing *h* for hamburger, *s* for salami, *t* for tuna, *c* for coffee, and *m* for milk. Then the above sentence showing the cross product simply becomes

$$\{h, s, t\} \times \{c, m\} = \{(h, c), (h, m), (s, c), (s, m), (t, c), (t, m)\}.$$

Using this example as a guide, we can now write a definition of the cross product of any two sets:

> *The cross product of two sets is the set of all possible pairs whose first member comes from the first set and whose second member comes from the second set. If the first set is named "A," and if the second set is named "B," then the cross product is named "A × B."*

Several comments about this definition are in order.

1. The symbol "×" is read "cross." Thus, "$A \times B$" is read "A cross B." The cross "×" does not denote ordinary multiplication. Ordinary multiplication applies to numbers, not sets.

2. The pairs in the cross product are *ordered* pairs. That is, of the two members of the pair, one comes first, the other second. In our example, the pair (h, c) is an element of the cross product $\{h, s, t\} \times \{c, m\}$, but the pair (c, h) is not. Of course, we could have formed the reverse cross product $\{c, m\} \times \{h, s, t\}$. This would be made of beverage-sandwich pairs rather than sandwich-beverage pairs. Then (c, h) would be an element of this *new* cross product.

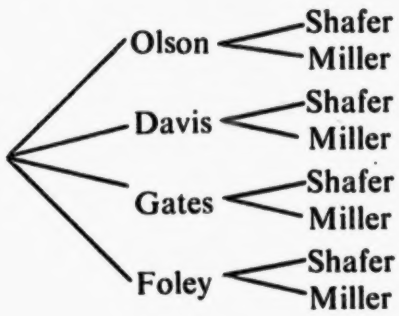
$$\{c, m\} \times \{h, s, t\} = \{(c, h), (c, s), (c, t), (m, h), (m, s), (m, t)\}.$$

3. Each pair in the cross product is considered a single element. Thus the set $\{(h, c), (h, m), (s, c), (s, m), (t, c), (t, m)\}$ has 6 elements.

Before developing multiplication by means of cross products, we present some exercises on cross products. Work these exercises before proceeding.

EXAMPLE: If a baseball team has 4 pitchers (Olson, Davis, Gates, and Foley) and 2 catchers (Shafer and Miller), what are all the pitcher-catcher combinations?

SOLUTION: There are 4 pitchers from which to choose. For each choice of a pitcher, there are 2 choices for the catcher. All the possible choices can be pictured on a "tree" diagram:



So all the pitcher-catcher combinations are as follows:

(Olson, Shafer)
(Olson, Miller)
(Davis, Shafer)
(Davis, Miller)
(Gates, Shafer)
(Gates, Miller)
(Foley, Shafer)
(Foley, Miller)

Exercise Set 1

1. Imagine an election in which there are 4 candidates for governor (call them a, b, c, and d) and 3 candidates for lieutenant governor (call them e, f, and g). List all the possible combinations of candidates from which the voters can choose.

2. Given the sets $\{a, b\}$ and $\{r, s, t, u\}$, list the members of the cross product $\{a, b\} \times \{r, s, t, u\}$.

3. If set $A = \{x, y\}$ and set $B = \{r, s, t\}$, list the members of the set $A \times B$. List the members of the set $B \times A$. Is $A \times B$ the same set as $B \times A$? Is $A \times B$ equivalent to $B \times A$?

4. In problem 3, what is $n(A)$? $n(A \times B)$? $n(B \times A)$?

Problem 1 in the exercises above asks you, in effect, to form the cross product of set $\{a, b, c, d\}$ and set $\{e, f, g\}$. Your answer should be

$$\{(a, e), (a, f), (a, g), (b, e), (b, f), (b, g), (c, e), (c, f), (c, g), (d, e), (d, f), (d, g)\}.$$

Counting the elements of each of these sets, we find that the cross product of a set of 4 elements and a set of 3 elements has 12 elements. It is immaterial what objects are denoted by a, b, c, d, e, f, or g. The cross product of *any* set of 4 elements with *any* set of 3 elements will always contain 12 elements. This observation enables us to assign a meaning to "4 × 3," using sets; 4 × 3 is simply the number of elements in the cross product of a set of 4 elements and a set of 3 elements, thus making $4 \times 3 = 12$.

Problem 2 above asks you to form the cross product of a set of 2 elements and a set of 4 elements. That cross product contains 2 × 4 elements. By counting we find that this cross product has 8 members, thus making 2 × 4 = 8.

Cross products helped us to determine the number 12 from 4 and 3, and 8 from 2 and 4. So we have a way of determining a number called a "product" from a pair of given numbers. Understanding what is meant by *product* is the key to understanding multiplication.

> *If a and b are whole numbers, let A and B be sets such that $n(A) = a$ and $n(B) = b$. The* product *of a and b, denoted by "$a \times b$," is $n(A \times B)$, that is, the number of elements in set $A \times B$. (We often write "ab" instead of "$a \times b$"; and we call a and b* factors *of ab.)*

The product $a \times b$ does not depend on the nature of the elements comprising set A, nor on the nature of the elements comprising set B.

The assignment of a product to a pair of numbers is essentially what we mean by *multiplication*.

How are products found? What is the product of 3 and 5, for example? According to the meaning we have just given to product, we should find a set containing 3 elements and a set containing 5 elements. (Any sets of 3 and 5 will do.) Then we should consider the cross product of these sets and count the elements of the cross-product set. The number of elements in the cross-product set will be 3 × 5, the product of 3 and 5. Accordingly, let us choose the sets

$$R = \{x, y, z\}.$$
$$S = \{a, e, i, o, u\}.$$

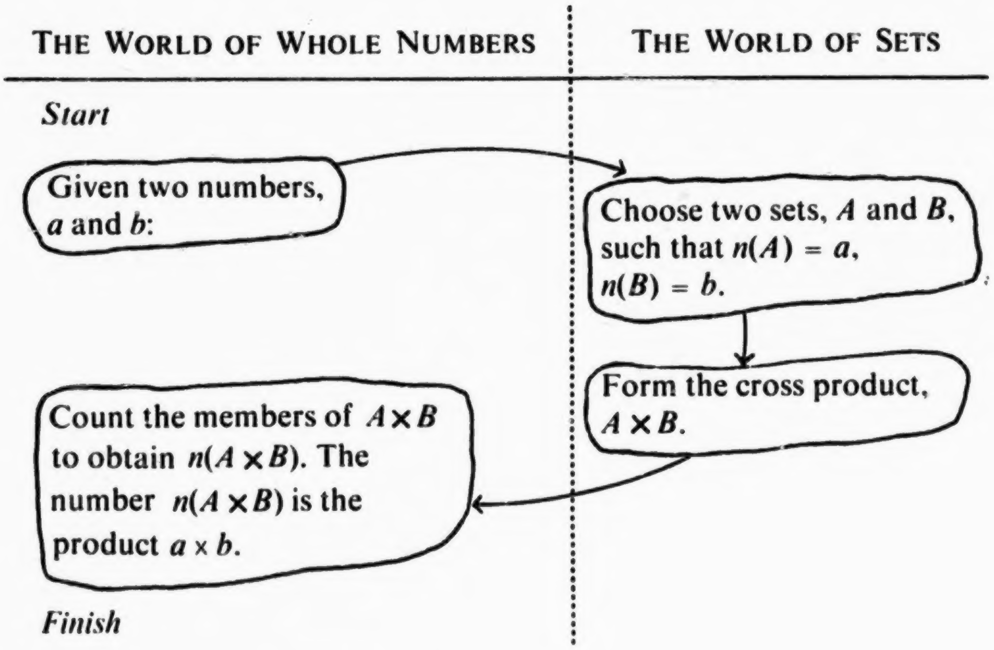
Since $n(R) = 3$ and $n(S) = 5$, we have the kind of sets we want. Now,

$$\begin{aligned} R \times S = \{&(x, a), (x, e), (x, i), (x, o), (x, u),\\ &(y, a), (y, e), (y, i), (y, o), (y, u),\\ &(z, a), (z, e), (z, i), (z, o), (z, u)\}. \end{aligned}$$

Counting shows that $n(R \times S) = 15$. *Therefore*, 3 × 5 = 15. We have *proved* that 15 is the product of 3 and 5. The proof rests upon the meaning we have given to product.

We have talked at length about both sets and numbers. However, multiplication itself deals only with numbers: with every pair of numbers multiplication associates their product, a number. The role of sets is to provide a means by which a product can be computed.

The scheme whereby we use sets to obtain a product may be diagramed as follows:


THE WORLD OF WHOLE NUMBERS
THE WORLD OF SETS
Start
Given two numbers, a and b:
Choose two sets, A and B, such that $n(A) = a$, $n(B) = b$.
Form the cross product, $A \times B$.
Count the members of $A \times B$ to obtain $n(A \times B)$. The number $n(A \times B)$ is the product $a \times b$.
Finish

You see that multiplication begins and ends with numbers. This completes our discussion of one way in which sets may be used to explain multiplication. After the next exercise set, a second way will be presented.

EXAMPLE: Mr. McCarthy travels from New York City to Chicago by either airplane, railroad, or bus. He travels from Chicago to Milwaukee by either bus or car. In how many ways can he travel from New York City to Milwaukee?

SOLUTION: For each of the three ways from New York City to Chicago, Mr. McCarthy can choose one of two ways from Chicago to Milwaukee. So each trip is an ordered pair. The set of these ordered pairs is the following:

{(airplane, bus), (airplane, car), (railroad, bus),
(railroad, car), (bus, bus), (bus, car)}

So Mr. McCarthy can travel from New York City to Milwaukee in six ways.

Exercise Set 2

1. A girl owns a red blouse, a white blouse, a brown skirt, a black skirt, and a white skirt. How many combinations of blouse and skirt can she wear? What are they?

2. What *multiplication* sentence should accompany this sentence?

$$\{a, b\} \times \{x, y\} = \{(a, x), (a, y), (b, x), (b, y)\}.$$

3. If $n(A) = 4$ and $n(B) = 5$, what is $n(A \times B)$?

UNION OF DISJOINT EQUIVALENT SETS

Unions of disjoint sets are used to develop a meaning for addition of whole numbers. Let us recall that—

> *The* union *of two sets is the set of elements in either or both of the two sets. The union of sets A and B is named "A ∪ B." More generally, the union of any collection of sets is the set consisting of all those elements which are members of at least one of the sets in the given collection. (The union of sets A, B, and C may be named "A ∪ B ∪ C.")*
>
> *Two sets are* disjoint *if and only if they have no elements in common.*
>
> *Two sets are* equivalent *if and only if they can be matched (by a one-to-one correspondence). Equivalent sets have the same number of elements. If two sets are not equivalent, then they do not have the same number of elements.*

If we form the union of several sets, every two of which are disjoint and equivalent, then we have a special situation that deserves attention.

Where might such a situation arise? Consider the following question:

> If a flock of chickens lay 6 eggs every day for 7 days, how many eggs are produced altogether?

Analyzed in terms of sets, the question states that a set of 6 eggs is produced every day for 7 days. The union of these 7 sets, all disjoint, is the set of eggs obtained after the 7 days have passed. So the total number of eggs is the *sum* of the numbers of the daily sets of eggs. That is, the total number of eggs is

$$6 + 6 + 6 + 6 + 6 + 6 + 6,$$

or 42.

Such situations can always be analyzed using addition. However, it is clear in this example that the number of sets of 6 eggs, namely 7, is important. In some way, the numbers 6 and 7 determine their "product" 42.

Examples of the above type have always been used by teachers to introduce multiplication. They suggest another meaning we can give to product:

> *Let a and b be whole numbers. Choose a sets, disjoint from each other, with b elements in each. Then the number of elements in the union of the a sets is a × b, the product of a and b.*

How can a product such as 3 × 5 be computed if we use this approach? We should choose 3 sets, with 5 elements in each, making sure that each is disjoint from the others. We then consider the union of the sets and count the elements in the union. The number of members in the union will be 3 × 5, the product of 3 and 5. Accordingly, let us choose these sets:

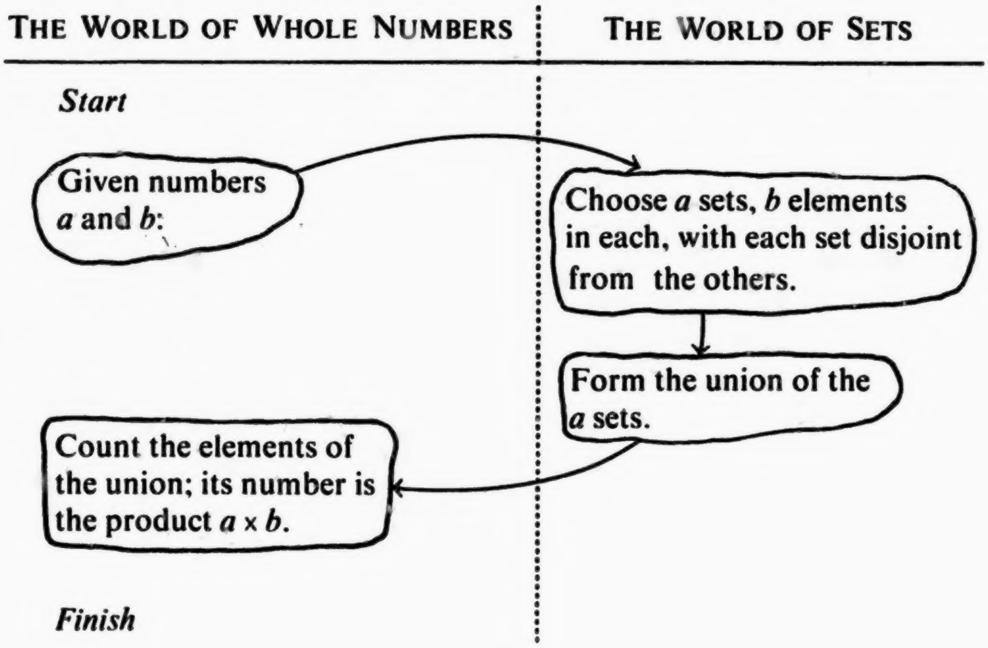
$$\begin{aligned} E &= \{a, b, c, d, e\}. \\ F &= \{i, j, k, l, m\}. \\ G &= \{r, s, t, u, v\}. \end{aligned}$$

The union, $E \cup F \cup G$, is

$$\{a, b, c, d, e, i, j, k, l, m, r, s, t, u, v\}.$$

Counting shows the number of elements in this union to be 15. *Therefore*, 3 × 5 = 15.

As in the case of cross products, we have now provided a scheme for determining a product from any pair of whole numbers. Again, the sole role played by sets is to provide a means by which a product can be computed from its factors. This scheme can be diagramed as follows:


THE WORLD OF WHOLE NUMBERS
THE WORLD OF SETS
Start
Given numbers a and b:
Choose a sets, b elements in each, with each set disjoint from the others.
Form the union of the a sets.
Count the elements of the union; its number is the product a × b.
Finish

Again we have seen that multiplication begins and ends with numbers.

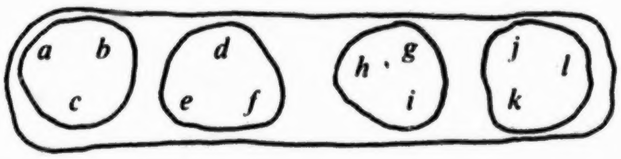
This completes the discussion of a second way in which sets can be used to develop multiplication. The next section will show that—

1. The two methods yield the same product for any pair of factors.
2. Both methods are useful in the classroom.

Exercise Set 3

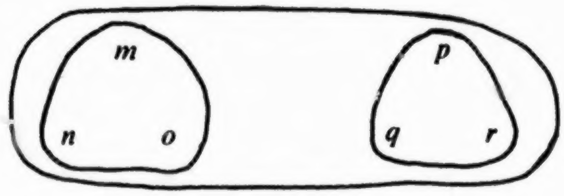
1. What multiplication facts do these pictures illustrate?

a.



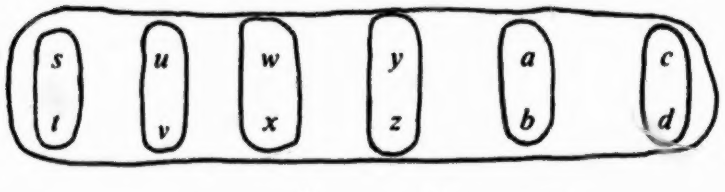
4 × ______ = ______.

b.



______ × ______ = ______.

c.



______ × ______ = ______.

2. Choose some disjoint equivalent sets to illustrate the fact that 2 × 4 = 8.

3. What multiplication sentence is equivalent to each of these addition sentences?

a. 3 + 3 + 3 + 3 + 3 + 3 = 18. ____ × 3 = ____.

b. 1 + 1 + 1 + 1 = 4. ____ × ____ = ____.

c. 0 + 0 + 0 = 0. 3 × ____ = ____.

d. 6 + 6 + 6 + 6 + 6 + 6 = 36. ____ × ____ = ____.

e. 2 + 2 = 4. ____ × ____ = ____.

4. In the following problem should we find the *sum* or the *product* of the two numbers? Why?

John bought 3 quarts of milk at 24¢ each. How much money did he spend?

TEACHING MULTIPLICATION

Both approaches to multiplication, through cross products and through unions of disjoint equivalent sets, should be presented to children. Why?

Here are two problems that a child might be asked to solve in the third or fourth grade:

(1) Jim planted 3 rows of corn, using 8 seeds in each row. How many seeds did he plant?

(2) If 3 children play trumpet and 8 other children play piano, how many trumpet-piano duos can be formed?

Each of these problems is solved by multiplying 3 and 8 to obtain 24. But each requires its own kind of thought process. A child will probably see the problems as entirely different, yet children need to learn to recognize both as multiplication problems. The two approaches we have discussed are appropriate to these two types of problems.

However, although these are two *approaches* to multiplication of whole numbers, they do not give different products. We present, as evidence that both yield the same result, a simple problem analyzed through both approaches.

PROBLEM: Compute the product of 3 and 4.

CROSS PRODUCT

Choose two sets, one of 3 elements, the other of 4 elements. The sets $\{a, b, c\}$ and $\{w, x, y, z\}$ will do.

Arrange the elements of these sets vertically and horizontally in a table and form the cross product:

	w	x	y	z
a	a, w	a, x	a, y	a, z
b	b, w	b, x	b, y	b, z
c	c, w	c, x	c, y	c, z

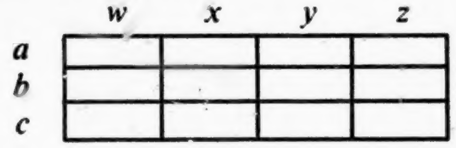
UNION OF DISJOINT EQUIVALENT SETS

Choose three sets, disjoint from each other, with 4 elements in each. The sets $\{a, b, c, d\}$, $\{i, j, k, l\}$, and $\{m, n, o, p\}$ will do.

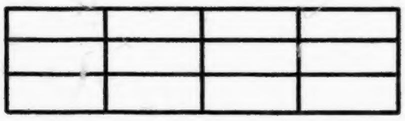
Arrange the elements of these sets in horizontal rows, in a table, and form their union:

a	b	c	d
i	j	k	l
m	n	o	

Observe that to count this cross product, we need only count the places in the table:

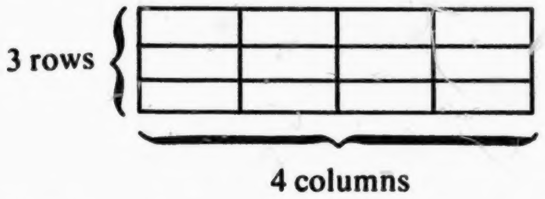


Observe that to count this union, we need only count the places in the table:



Both these approaches lead to the same result: a rectangular *array* of places that we are to count to find a product.

An array that has 3 rows and 4 columns (rows are horizontal, columns are vertical) is called a "3-by-4 array."



Thus, a 3-by-4 array represents the product 3 × 4, *regardless of which approach to product we use*. Of course, any 3 × 4 array will do:

☆ ☆ ☆ ☆   □ □ □ □   x x x x
☆ ☆ ☆ ☆ or □ □ □ □ or x x x x   etc.
☆ ☆ ☆ ☆   □ □ □ □   x x x x

Arrays are well suited to introducing multiplication to younger children. Later on, when problems like (2) on page 55 come up, an explicit presentation of cross products will also be helpful. When problems like (1) on page 55 come up, unions of disjoint equivalent sets can be presented.

In this text, we shall find arrays useful for deriving various properties of multiplication.

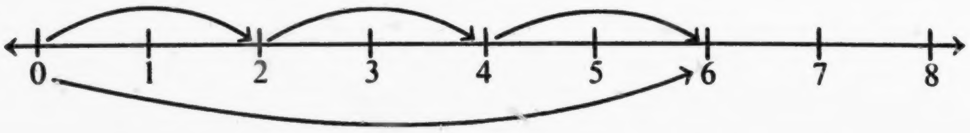
EXAMPLE: Compute the product of 6 and 2 by means of an array.

SOLUTION: Draw an array with 6 rows and 2 columns:

o o
o o
o o
o o
o o
o o

There are 12 elements in this 6-by-2 array, so 6 × 2 = 12.

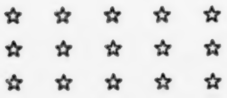
Still another approach to multiplication uses the number line. The expression "3 x 2" is interpreted as 3 "jumps" of 2 units each.



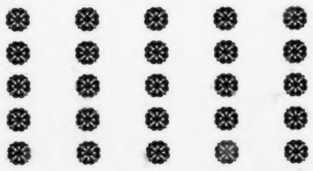
This interpretation is evidently equivalent to repeated addition on the number line.

Exercise Set 4

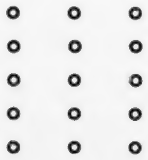
1. Write a multiplication sentence for each of these arrays:

a.


____x____=____.

b.


____x____=____.

c.


____x____=____.

2. Draw an array for each of these sentences:

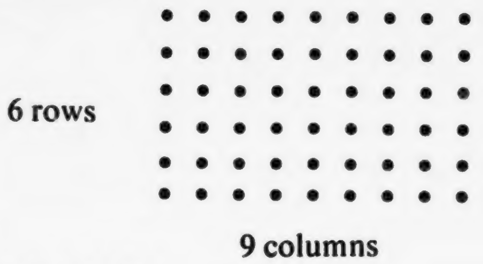
a. 3 x 2 = 6. **b.** 6 x 1 = 6. **c.** 4 x 7 = 28.

3. Use a number line to depict the following products:

a. 3 x 4 **b.** 4 x 3 **c.** 5 x 1 **d.** 1 x 5

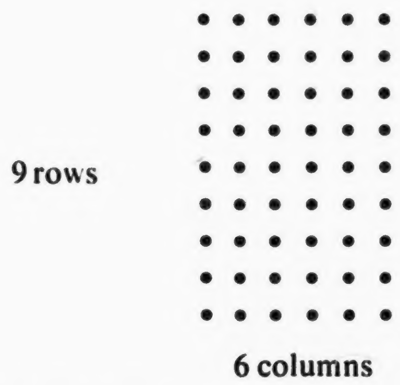
PROPERTIES OF MULTIPLICATION

Consider this 6-by-9 array of dots:



This array contains 6 x 9 dots.

Now consider a 9-by-6 array of dots:

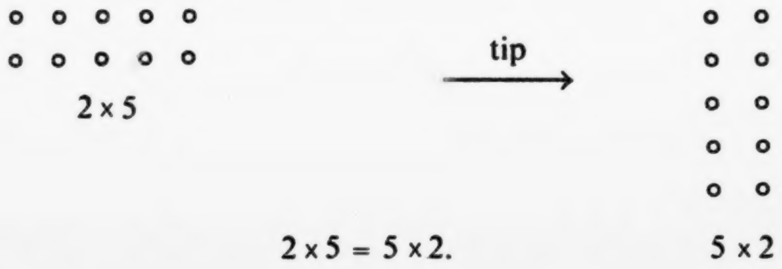


This array contains 9 × 6 dots.

Of the two arrays above, either could be "tipped" to coincide with the other. They contain the same number of dots. So

$$6 \times 9 = 9 \times 6.$$

There is nothing special about this example. Any rectangular array can be tipped so that its rows become columns and its columns become rows without affecting the number of elements.



$$2 \times 5 = 5 \times 2.$$

These examples are instances of a general property of multiplication, the *commutative* property:

For all whole numbers a and b, $a \times b = b \times a$.

To express the commutative property of multiplication in a form suitable for young children, "frames" are valuable. Children may be encouraged to fill in the frames in sentences like these:

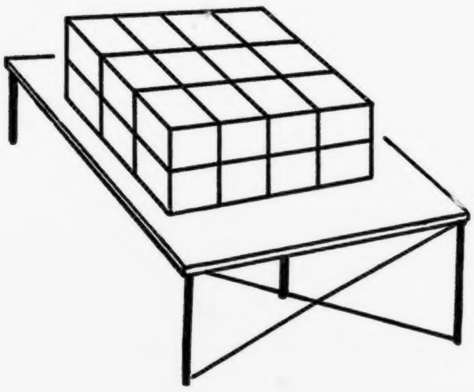
$$8 \times \square = \square \times 8. \qquad \triangle \times 7 = 7 \times \triangle.$$
$$\triangle \times \square = \square \times \triangle.$$

The same number is to be used with a particular frame wherever that frame appears in a single sentence. Children will discover that any numbers make $\triangle \times \square = \square \times \triangle$ a true statement:

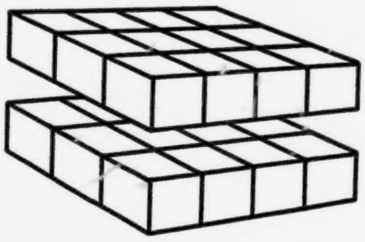
$$\overset{3}{\triangle} \times \boxed{6} = \boxed{6} \times \overset{3}{\triangle}; \qquad \overset{7}{\triangle} \times \boxed{1} = \boxed{1} \times \overset{7}{\triangle};$$
$$\overset{4}{\triangle} \times \boxed{5} = \boxed{5} \times \overset{4}{\triangle}; \text{ etc.}$$

The most obvious advantage of introducing the commutative property of multiplication in elementary school is that it reduces considerably the number of multiplication facts a child must learn. A less obvious, but far more important, advantage is that this commutative property is part of the basic structure of the whole-number system, and of other number systems.

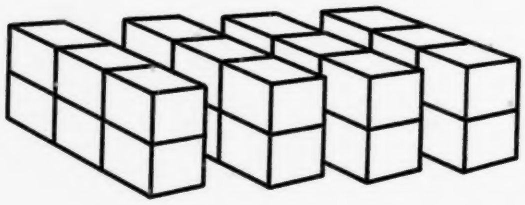
Another important property of multiplication can be demonstrated by looking at a rectangular stack of blocks in two different ways.



Think of this stack as being composed of horizontal layers or of vertical slabs:



The number of blocks
is 2 x (3 x 4).



The number of blocks is
4 x (2 x 3) or (2 x 3) x 4
because multiplication is commutative.

So the number of blocks in the stack can be named in two ways:

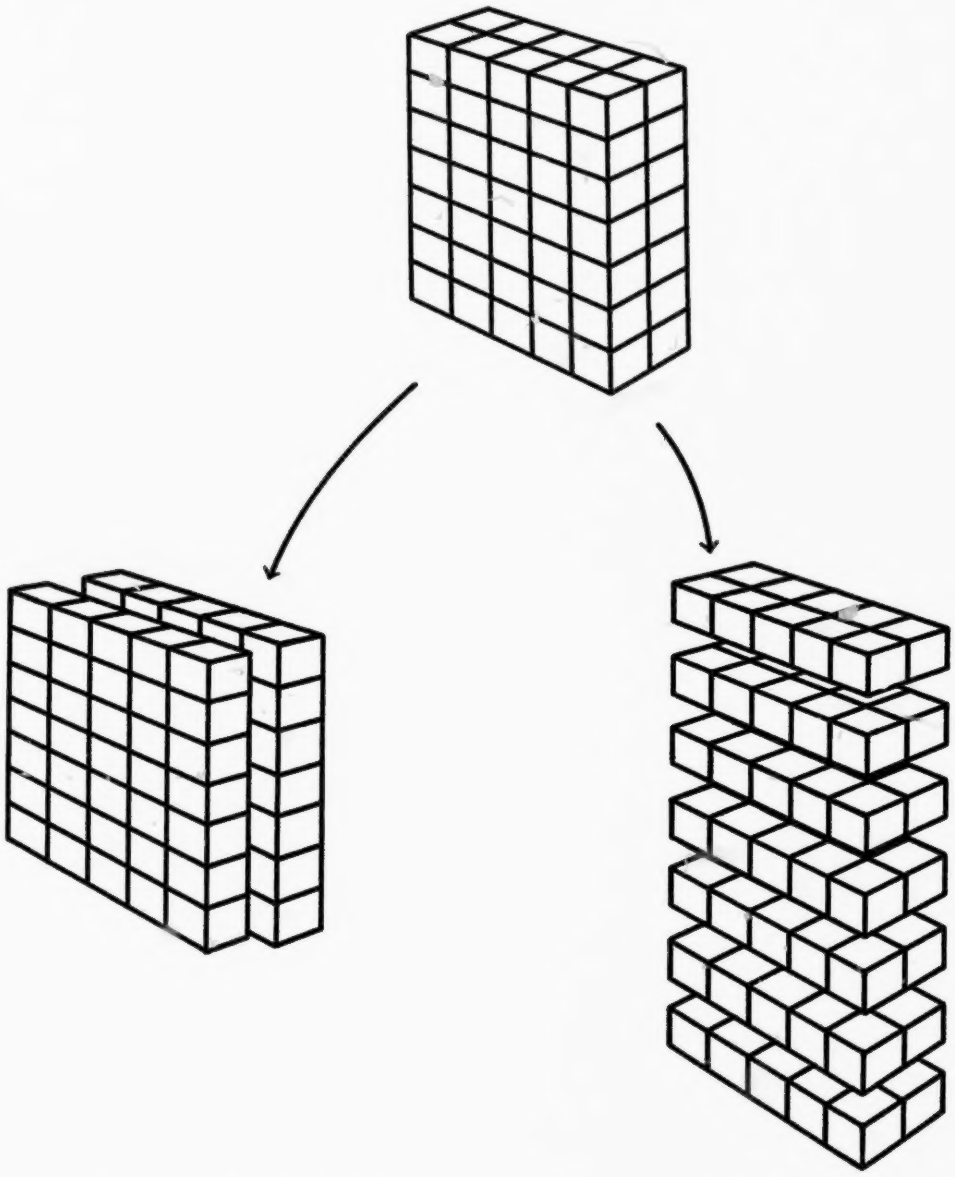
$$2 \times (3 \times 4) \quad \text{and} \quad (2 \times 3) \times 4.$$

Therefore,

$$2 \times (3 \times 4) = (2 \times 3) \times 4.$$

Evidently, it makes no difference when we multiply whether we associate the 3 with the 4 or with the 2.

A stack of blocks, or 3-dimensional array, of any size, can be viewed in these two ways. Consider, as a second example, the pictures below:



The number of blocks
is $2 \times (7 \times 5)$
or $(7 \times 5) \times 2$
because multiplication is commutative.

The number of blocks
is $7 \times (5 \times 2)$.

So $(7 \times 5) \times 2 = 7 \times (5 \times 2)$.

These examples are not sufficient to prove anything, but they suggest a generalization. This generalization is the *associative* property of multiplication:

For all whole numbers a, b, and c, $a \times (b \times c) = (a \times b) \times c$.

Thus, like addition, multiplication is both commutative and associative.

The associative property of multiplication is important both as a building block in the structure of whole numbers and as an aid in computation. If we want to compute the product of 2 and 30, for example, we may reason as follows:

2×30	$= 2 \times (3 \times 10)$	because $30 = 3 \times 10$
	$= (2 \times 3) \times 10$	because multiplication is associative
	$= 6 \times 10$	because $2 \times 3 = 6$
	$= 60$	because $6 \times 10 = 60$.

Or suppose we want to compute the product $(63 \times 4) \times 25$. An application of the associative property makes it easy:

$$\begin{aligned}(63 \times 4) \times 25 &= 63 \times (4 \times 25)\\ &= 63 \times 100\\ &= 6{,}300.\end{aligned}$$

Often, the commutative and associative properties are both used in a computation. To compute the product 24×5, for example, we might reason as follows:

24×5	$= (2 \times 12) \times 5$	because $24 = 2 \times 12$
	$= (12 \times 2) \times 5$	because multiplication is commutative
	$= 12 \times (2 \times 5)$	because multiplication is associative
	$= 12 \times 10$	because $2 \times 5 = 10$
	$= 120$	because twelve tens $= 120$.

The commutative and associative properties, used together, make it possible to rearrange factors in any order or in any grouping without affecting the product. No parentheses are needed in an expression for a product. For example, the expression $2 \times 9 \times 5 \times 7$ is not ambiguous;

the product can easily be computed by first multiplying 2 by 5, then 9 by 7, then 10 by 63, thus obtaining the following equalities:

$$\begin{aligned} 2 \times 5 &= 10. \\ 9 \times 7 &= 63. \\ 10 \times 63 &= 630. \end{aligned}$$

Exercise Set 5

1. Use the commutative property of multiplication and complete to make true statements:

a. $\square \times 8 = 8 \times 9.$

b. $4 \times \triangle = 17 \times 4.$

c. $67 \times \square = 87 \times 67.$

d. $\square \times \triangle = 17 \times 13.$

e. $(\square \times 6) \times 7 = (6 \times 9) \times 7.$

f. $(16 \times 35) \times \square = 12 \times (16 \times 35).$

2. Use the associative property of multiplication and complete to make true statements:

a. $(\square \times 6) \times 7 = 5 \times (6 \times 7).$

b. $(8 \times \triangle) \times 9 = 8 \times (9 \times 9).$

c. $16 \times (\square \times 4) = (16 \times 8) \times \triangle.$

3. Compute these products quickly with the help of the commutative and associative properties:

a. (67 x 50) x 2 = ______.

b. (5 x 13) x 2 = ______.

c. 8 x 700 = ______.

4. Complete these to make true statements:

a. (16 − 9) − 3 = ______ − 3 = ______.
16 − (9 − 3) = 16 − ______ = ______.

b. 18 − (7 − 4) = 18 − ______ = ______.
(18 − 7) − 4 = ______ − 4 = ______.

c. Do you think *subtraction* is associative?

THE NUMBERS 0 AND 1

The numbers 0 and 1 play special roles in multiplication. What kind of array represents the product 5 x 1?

A 5-by-1 array:

$$5 \text{ rows} \left\{ \underbrace{\begin{array}{c} \star \\ \star \\ \star \\ \star \\ \star \end{array}}_{\text{1 column}} \right.$$

What kind of array represents the product 1 x 5?

A 1-by-5 array:

$$\text{1 row} \left\{ \underbrace{\star \quad \star \quad \star \quad \star \quad \star}_{\text{5 columns}} \right.$$

Counting these arrays shows that $5 \times 1 = 5$ and $1 \times 5 = 5$. We could have computed 6×1, 1×6, 3×1, 1×3, etc., in the same way. Evidently, when 1 is used as a factor, the product is the other factor. In general terms,

Whenever b is a whole number,
$b \times 1 = b$ *and* $1 \times b = b$.

This states what is often called the *multiplicative property of 1*. The number 1 is called the *identity element for multiplication*. It is also sometimes called the *neutral* element for multiplication.

Notice that 1 plays the same role in multiplication that 0 plays in addition.

$$3 \times 1 = 3; \qquad 1 \times 75 = 75;$$
$$3 + 0 = 3; \qquad 0 + 75 = 75; \qquad \text{etc.}$$

The multiplicative property of 1 is another of the important building blocks of the whole-number system. It is particularly important when computing with fractions. However, there is value in knowing that 19 of the usual 100 multiplication facts have 1 as a factor.

The number 0 plays a role in multiplication that has no counterpart in addition.

If we attempt to illustrate the product 5×0 with an array, we need an array with 5 rows and 0 columns. Since this is difficult to visualize, let us use the cross product to analyze 5×0. We choose a set of 5 elements, say $\{a, b, c, d, e\}$, and a set of 0 elements– the empty set, $\{\ \}$. We now wish to form pairs with one member from $\{a, b, c, d, e\}$ and the other member from $\{\ \}$. But there are no elements in $\{\ \}$ to use; so no pairs can be formed. Therefore, the cross product of $\{a, b, c, d, e\}$ and $\{\ \}$ will have no elements.

$$\{a, b, c, d, e\} \times \{\ \} = \{\ \}.$$

What can we conclude about 5×0? If 5×0 is to be the number of the cross product we formed, we must have

$$5 \times 0 = 0.$$

If we had analyzed 0×5, the result would have been the same:

$$\{\ \} \times \{a, b, c, d, e\} = \{\ \}, \text{ so}$$

$$0 \times 5 = 0.$$

In fact, the product of 0 and *any* number can be analyzed this way. We conclude the following:

Whenever b is a whole number,
$b \times 0 = 0$ *and* $0 \times b = 0.$

Children can be introduced to this *multiplicative property of* 0 by completing sentences such as these:

$\square \times 0 = 0; \quad 3 \times \square = \square; \quad \square \times \triangle = 0;$ etc.

AN IMPORTANT MULTIPLICATION PROPERTY INVOLVING ZERO

Under what circumstances could a product of whole numbers be zero? If a and b are whole numbers and we are told that their product, $a \times b$, is zero, what can we say about the factors a and b?

In terms of arrays we interpret a product $a \times b$ to be the number of elements in an array having a rows and b columns. If neither a nor b is zero, such an array will surely have at least one member. That is, if $a \neq 0$ and $b \neq 0$, then $a \times b \neq 0$. It follows logically that the only way $a \times b$ could be zero would be if either a or b (or possibly both) were zero.

If $a \times b = 0$, then $a = 0$ or $b = 0$ (or both a and b equal zero). This important principle applies not only to whole numbers but to larger classes of numbers as well (for example, fractional numbers). It is used extensively in solving equations.

Exercise Set 6

1. Complete these to make true statements:

a. $7 \times \square = 0.$

b. $0 \times 6 = \square.$

c. $75 \times \square = 75.$

d. $\square \times 1 = 10.$

e. $1 \times 0 = \square.$

f. $\square \times 7 = \square.$

2. What numbers make these true statements?

a. $\square \times 1 = \square.$

b. $0 \times \square = 0.$

c. $\square \times \square = \square.$

3. Compute these products quickly. Which properties did you use?

a. 3 x 7 x 85 x 0 x 96 = ______.

b. 576 x 1 = ______.

c. (75 – 75) x 37 = ______.

4. If the commutative property and the multiplicative properties of 1 and 0 are used, which facts in this table need to be memorized?

x	0	1	2	3	4	5	6	7	8	9
0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8	9
2	0	2	4	6	8	10	12	14	16	18
3	0	3	6	9	12	15	18	21	24	27
4	0	4	8	12	16	20	24	28	32	36
5	0	5	10	15	20	25	30	35	40	45
6	0	6	12	18	24	30	36	42	48	54
7	0	7	14	21	28	35	42	49	56	63
8	0	8	16	24	32	40	48	56	64	72
9	0	9	18	27	36	45	54	63	72	81

SUMMARY

Some children's language is suggested below for some of the ideas we have presented.

ADULT LANGUAGE	CHILD LANGUAGE
Ordered pairs	Pairs
Union of 3 sets of disjoint equivalent sets of 5 elements each	Union of 3 sets of 5
Commutative property	Order property
Associative property	Grouping property
Multiplicative property of 1	Using 1 as a factor (or, multiplying by 1 gives the same number)
Multiplicative property of 0	Using 0 as a factor (or, zero times any number gives zero)

The approach and language of this lesson have been adult. Children need not be so careful about definitions and properties, nor need they use such exact language

It is important to avoid teaching mathematics as a vocabulary exercise. The idea of multiplication and its properties can be presented to children at their level.

SUBTRACTION

$$8-5=3$$

1. How can subtraction be explained—
  ***a)* Through the use of sets?**
  ***b)* Using ideas of addition?**
2. What does an expression such as "5 − 3" mean?

Have you ever given your pupils subtraction exercises and told them to check by adding? Perhaps one of the exercises was

$$\begin{array}{r} 46 \\ \underline{-\ 19} \end{array}$$

If a pupil thought the answer was 27, he was supposed to check by adding 27 to 19. He hoped to obtain 46 if his answer to the subtraction exercise was correct.

$$\begin{array}{r} 27 \\ \underline{+\ 19} \\ 46 \end{array}$$

That is, he added his answer (27) to the smaller of the numbers he was given (19) and hoped to obtain the larger (46).

Evidently subtraction has some relationship to addition. What is this relationship? How should it be presented to children? What are the consequences of this relationship? This chapter will explore these and other questions.

The relationship just mentioned concerns the very meaning of subtraction. We teach children that "seven minus three equals four," but we also need to teach then *why*. If a child says, "Seven minus three equals five," we must be able to show him why he is wrong. In other words, we must convey the *meaning* of subtraction to children.

EXPLAINING SUBTRACTION

Let us recall briefly how addition is usually defined in the new mathematics programs. Pupils are first shown how to find the union of two disjoint sets. Later they learn that the *number* of elements of such a union is called the *sum* of the numbers of the two sets. Certain properties of addition (such as the commutative property) are then deduced from the definition.

Subtraction may be based on sets also, by using the notion of a subset. However, even when subtraction is defined in terms of sets, it quickly becomes apparent that subtraction is related to addition. Thus the meaning of subtraction can be approached in two ways: (1) in terms of sets; (2) in terms of addition. As an illustration of the approach using sets, let us consider the following situation. Mary's brothers are Mike, John, Max, Bill, George, Frank, and Don. Designating the set of Mary's brothers by A, we have

$$A = \{\text{Mike, John, Max, Bill, George, Frank, Don}\}.$$

Let us also consider the set of Mary's brothers whose names, as given here, begin with "M"; call this set B.

$$B = \{\text{Mike, Max}\}.$$

Since all the elements of set B are also elements of set A, we say set B is a *subset* of set A.

Those brothers whose names do not begin with "M" form another subset of A; call this subset C.

$$C = \{\text{John, Bill, George, Frank, Don}\}.$$

Now let us ask a simple numerical question such as might be appropriate for early grades:

> Mary has 7 brothers; only 2 of them have names beginning with "M." How many other brothers does Mary have?

If a pupil lists sets A, B, and C as above, then to answer the question he should count set C, the brothers whose names do not begin with "M."

Evidently, this example can be analyzed from two points of view:

1. Set A and set B are given. Set B is a subset of set A. What is the subset of A whose elements are not in B? Set C.

2. A set of seven elements and one of its subsets consisting of two elements are given. How many elements of the set of seven elements are *not* in this subset of two elements? Five.

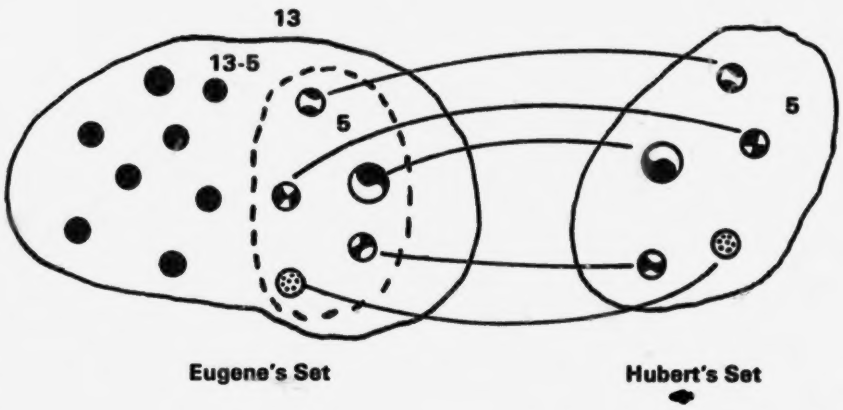
We call this type of problem a *subtraction* problem. In it, we represent the number of elements that are *not* in the given subset by "$7 - 2$" (read "seven minus two"), and we refer to this number as the "difference of 7 and 2." Because this difference is 5, we write "$7 - 2 = 5$." (Children might, for example, remove two blocks from a set of seven blocks, then count the blocks remaining. They would write 7 − 2, or 5, to express the result.) We say that subtraction assigns to the pair of numbers 7 and 2 the difference 7 − 2, or 5.

Any problem about objects "remaining" or being "taken away" or "left over" can be solved by subtraction, as indicated here.

One kind of problem often called a "comparison" problem fits the subtraction pattern less obviously. Here is an example:

> Eugene has 13 marbles; Hubert has 5 marbles. How many more than Hubert does Eugene have?

In this situation, Hubert's 5 marbles are not a subset of Eugene's 13 marbles, so we can't simply seek the remaining subset. But the required procedure is obvious (though not to a child, perhaps): We match Hubert's 5 marbles with 5 of Eugene's marbles. We then seek the remaining subset of Eugene's marbles.



The number of the remaining set is 13 − 5. This tells "how many more" marbles Eugene has.

Using sets, we may formally state a definition of $a - b$, the difference of a and b, as follows:

> *If A is a set that contains a elements and B is a subset of A that contains b elements, then $a - b$ is the number of the subset of elements of A that are* not *in B.*

The difference $a - b$ does not depend on the selection of sets A and B, as long as these sets fulfill the specified requirements. One restriction on the numbers involved in subtraction arises from this definition. This definition says that in order to subtract b from a, b must be the number of a subset of a set of a elements. Clearly, b *cannot* be greater than a. So expressions like 3 − 5, 17 − 18, etc., make no sense at this stage.

Exercise Set 1

1. For each of these exercises, answer these questions:

Is B a subset of A?
If so, what is the subset of A composed of elements *not* in B?

a. $A = \{a, e, f, i, j, o, p, u\}$.
$B = \{a, e, i, o, u\}$.

b. $A =$ {red, white, blue, green}.
$B =$ {green}.

c. $A =$ {California, Oregon, Washington, Maine, Florida}.
$B =$ {Maine, Florida}.

d. $A = \{x, y, z\}$.
$B = \{x, y, z\}$.

e. $A = \{△, □, ○\}$.
$B = \{\ \}$.

2. For each of the exercises above, write a subtraction sentence. The subtraction sentence for 1a is $8 - 5 = 3$ because, given a set of 8 elements and a subset of 5 elements, the remaining subset has 3 elements.

3. If B is a subset of A, as above, and if C is the subset of A composed of elements *not* in B, then what set is $B \cup C$?

So far in our development of subtraction, we have not mentioned addition. Yet subtraction is often called the "inverse" of addition. Why?

In the examples in the exercises and the text above, we began with a set and a subset of that set; then we asked for the remaining subset. In Exercise 1a, the given set was

$$A = \{a, e, f, i, j, o, p, u\},$$

and the subset was

$$B = \{a, e, i, o, u\}.$$

The remaining set that we found was

$$C = \{f, j, p\}.$$

Now, since C is the set of elements of A that are *not* in B, the sets C and B are disjoint. We may form their union, $B \cup C$; clearly $B \cup C = A$. That is,

$$\{f, j, p\} \cup \{a, e, i, o, u\} = \{a, e, f, i, j, o, p, u\}.$$

Whenever we form a union of two disjoint sets, we have an addition situation. The union shown above implies the addition sentence

$$3 + 5 = 8.$$

The subtraction sentence in Exercise 1a is

$$8 - 5 = 3.$$

So addition is certainly related to subtraction.

But let us make that relationship more explicit by returning to the sets in Exercise 1a.

$$A = \{a, e, f, i, j, o, p, u\}.$$
$$B = \{a, e, i, o, u\}.$$

Instead of asking, "What is the subset of A whose elements are *not* in B?" we could ask, "What is the set, disjoint from B, that will correctly complete this sentence: $B \cup \square = A$?" The answer to each question is the same: $\{f, j, p\}$.

Now let us ask numerical questions about this example. We might ask, "What is the number of elements of the subset of A whose elements are not in B?" Or, we might ask the same question this way: "*What number added to 5 will give a sum of 8?*"

In other words, to find the difference of 8 and 5, we may complete the sentence $5 + \square = 8$. Thus the sentence $5 + \square = 8$ has the same meaning as the sentence $\square = 8 - 5$.

Numbers that are added are called *addends*. Completing a sentence such as $5 + \square = 8$ may be called "finding the missing (or unknown) addend." Thus, subtraction is sometimes called "the operation of finding the missing addend." In addition, we seek a sum of two given addends, while in subtraction we seek one of the addends of a given sum. This is why subtraction is sometimes called the inverse of addition.

If subtraction means finding the missing addend, what does $8 - 5$ mean? $8 - 5$ is the number which when added to 5 gives 8, namely 3. So $8 - 5 = 3$. The difference, 3, may be found by completing $5 + \square = 8$. In this sentence, 5 is often called the known or given addend and 8 is called the sum. Note that the expression "$8 - 5$," reading from left to right, first shows the sum 8, then minus, then the given addend 5.

Of course, if the known addend is greater than the desired sum, it will not be possible to find a suitable missing addend among the whole numbers. For example, the sentences $6 + \square = 2$ and $\square + 9 = 8$ cannot be completed with whole numbers. Thus, expressions such as $2 - 6$ and $8 - 9$ have no meaning when working within the set of whole numbers.

In this way we are led to the same restriction that we saw in our definition of subtraction using sets:

The known addend cannot be greater than the sum.

(By the time your students reach junior high school, however, they will find that there are numbers—namely, the negative numbers—that will suffice to complete sentences like $6 + \square = 2$.) The missing-addend approach to subtraction applies, therefore, to larger classes of numbers than does the approach using sets—for example, to a class that includes not only whole numbers but also negative numbers and fractional numbers.

We now use the missing-addend idea to describe subtraction as follows:

Subtraction assigns to the pair of whole numbers a and b the missing addend in the sentence

$$b + \square = a.$$

This missing addend is named "$a - b$." It is also called the *difference* of a and b. The expression "$a - b$" names a whole number only when b is not greater than a.

It follows that when we complete a sentence such as $\square + 6 = 19$, we have in effect "subtracted" 6 from 19, and we know that a name for the missing addend is "19 – 6." Since, from our knowledge of addition, $\boxed{13} + 6 = 19$, we get $13 = 19 - 6$.

If children are introduced to subtraction through finding missing addends, it is not absolutely necessary for them to memorize "subtraction facts." For example, if a child is asked to complete the sentence

$$11 - 4 = \square$$

(that is, asked to subtract 4 from 11), he should feel free to think

$$11 = \square + 4$$

or

$$11 = 4 + \square$$

and then to complete either sentence from his knowledge of addition facts. The missing addend that he finds, is the difference of 11 and 4, $11 - 4$. The child should learn that "eleven minus four equals seven" because "seven plus four equals eleven."

Can you now explain why addition can check subtraction?

Exercise Set 2

1. Write two subtraction sentences for each of these addition sentences.

a. $6 + 4 = 10$. $6 = 10 - 4$. $4 = 10 - 6$.

b. $8 + 1 = 9$. __________ __________

c. $12 + 0 = 12$. __________ __________

d. $4 + 14 = 18$. __________ __________

e. $154 + 67 = 221$. __________ __________

2. Write an addition sentence for each of these subtraction sentences.

a. $12 - 7 = 5$. $12 = 5 + 7$ (or $12 = 7 + 5$).

b. $6 - 6 = 0$. __________

c. $8 - 0 = 8$. __________

d. $10 - 5 = 5$. __________

e. $74 - 67 = 7$. __________

3. Convert each of these sentences to a subtraction sentence. Then complete the sentences.

a. $3 + \square = 12$. $\boxed{9} = 12 - 3$.

b. $\square + 6 = 7$. __________

c. $12 + \square = 12$. __________

d. $14 + \square = 15$. __________

e. $\square + 28 = 95$. __________

4. Convert each of these sentences to an addition sentence. Then complete both sentences.

a. $\square = 16 - 9$. $9 + \boxed{7} = 16$ (or $\boxed{7} + 9 = 16$).

b. $\square = 4 - 1$. __________

c. $6 - 2 = \square$. __________

d. $\square = 9 - 9$. __________

e. $\square = 75 - 72$. __________

PROPERTIES OF SUBTRACTION

In the new mathematics programs, children learn not only the meaning of addition and multiplication but also the properties of these mathematical operations. Two important properties of addition and multiplication are the associative property and the commutative property.

ASSOCIATIVE AND COMMUTATIVE PROPERTIES
OF MULTIPLICATION AND ADDITION

OPERATION	ASSOCIATIVE PROPERTY	COMMUTATIVE PROPERTY
Multiplication	For all whole numbers *a*, *b*, and *c*, $(a \times b) \times c = a \times (b \times c)$. *Example:* $(3 \times 6) \times 4 = 3 \times (6 \times 4)$.	For all whole numbers *a* and *b*, $a \times b = b \times a$. *Example:* $12 \times 7 = 7 \times 12$.
Addition.	For all whole numbers *a*, *b*, and *c*, $(a + b) + c = a + (b + c)$. *Example:* $(3 + 6) + 4 = 3 + (6 + 4)$.	For all whole numbers *a* and *b*, $a + b = b + a$. *Example:* $12 + 7 = 7 + 12$.

Does subtraction have these properties also? Let us consider two examples:

1. *Is subtraction commutative?* Does $8 - 3 = 3 - 8$? Clearly, $8 - 3 = 5$ because 5 correctly completes the sentence $3 + \square = 8$. But "3 − 8" is not a name for 5; in fact it is not a name for any whole number, since no whole number fits the sentence $8 + \square = 3$. So $8 - 3 \neq 3 - 8$. The symbol "≠" means "does not equal." (Using negative numbers, we would find that $3 - 8$ is -5, not 5, so that here, too, $8 - 3 \neq 3 - 8$.) Other such examples are easy to think of, but no more are necessary for our purpose. Subtraction would be commutative only if $a - b = b - a$ for *all* whole numbers *a* and *b*. The above example shows that there is at least one exception—that is, that $a - b$ does *not* equal $b - a$ for *all* whole numbers. By showing that $8 - 3$ is not equal to $3 - 8$, we have found an exception. So, *subtraction is not commutative.*

2. *Is subtraction associative?* For example, does $9 - (5 - 3)$ equal $(9 - 5) - 3$?

$$9 - (5 - 3) = 9 - 2 = 7,$$

but

$$(9 - 5) - 3 = 4 - 3 = 1.$$

So $9 - (5 - 3) \neq (9 - 5) - 3$. This exception shows that *subtraction is not associative*. (For subtraction to be associative, it would be necessary that $a - (b - c) = (a - b) - c$ for *all* whole numbers a, b, c.)

Exercise Set 3

1. Insert "=" or "≠," whichever applies, in each circle.

a. $3 - 2$ ○ $2 - 3$.

b. $(6 - 4) - 0$ ○ $6 - (4 - 0)$.

c. $8 \div 8$ ○ $8 \div 8$.

d. 86×74 ○ 74×86.

e. $6 - (4 - 1)$ ○ $(6 - 4) - 1$.

2. Insert parentheses to make each sentence true.

a. $8 - 4 - 1 = 5$.

b. $24 \div 6 \div 2 = 2$.

c. $12 - 7 - 0 = 5$.

d. $9 \times 4 \times 2 = 72$.

e. $2 \times 4 + 7 = 22$.

THE ROLE OF ZERO IN SUBTRACTION

You are already aware that the number 0 plays a special role in addition.

ADDITION PROPERTY OF 0

For every whole number a,

$$a + 0 = a \quad \text{and} \quad 0 + a = a.$$

Examples:

$$5 + 0 = 5.$$
$$0 + 16 = 16.$$

This special rule leads to some interesting facts about 0 in subtraction. These facts, although presented here in a somewhat abstract way, are best communicated to children through examples and exercises.

The addition property of zero leads to such sentences as $5 + 0 = 5$, $0 + 16 = 16$, $0 + 65 = 65$, $2 + 0 = 2$, etc. From each such addition sentence two subtraction sentences can be formed.

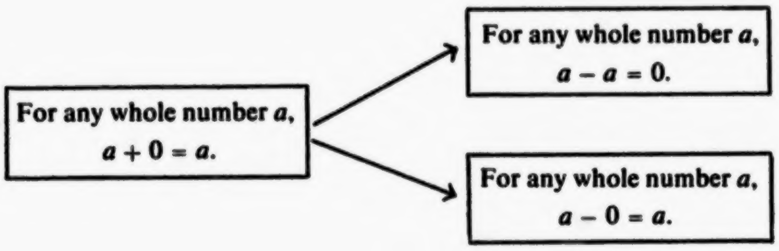
$5 + 0 = 5$ leads to $5 - 5 = 0$ and $5 - 0 = 5$.
$0 + 16 = 16$ leads to $16 - 16 = 0$ and $16 - 0 = 16$.
$0 + 65 = 65$ leads to $65 - 65 = 0$ and $65 - 0 = 65$.
$2 + 0 = 2$ leads to $2 - 2 = 0$ and $2 - 0 = 2$.

These results suggest the possibility of generalizing:

1. The fact that $5 - 5 = 0$, $16 - 16 = 0$, etc., suggests that "*any whole number subtracted from itself yields 0*."

2. The fact that $5 - 0 = 5$, $16 - 0 = 16$, etc., suggests that "*0 subtracted from any whole number yields that whole number*."

These generalizations constitute the role of 0 in subtraction; they can be proved from the addition property of 0.



For any whole number a, $a + 0 = a$.

For any whole number a, $a - a = 0$.

For any whole number a, $a - 0 = a$.

To summarize: What does $5 - 3$ mean? $5 - 3$ is the number that correctly completes the sentence

$$3 + \square = 5.$$

If from a sum of two addends one of the addends is subtracted, the difference is the remaining addend.

Subtraction is neither commutative nor associative.

Any number subtracted from itself yields 0.

0 subtracted from any number yields that number.

"SHIFTING OF TERMS" IN SUBTRACTION

Even after the meaning of subtraction is thoroughly understood, there still remains the practical problem of performing subtraction computations efficiently. In simple problems such as $11 - 4 = \square$ the missing-addend approach, namely

$$11 = 4 + \square,$$

can be used effectively by a child who knows his elementary addition facts.

However, in problems of even moderate complexity, a direct use of the missing-addend approach is often not practical. For example, it may not help a beginner very much if he tries to calculate 46 − 19 = □ by writing 46 = 19 + □. He learns how to break this problem up into simpler parts which he can handle with the knowledge already at his disposal. To accomplish this, he can use a subtraction algorithm* whereby he computes the difference:

$$\begin{array}{r} 46 \\ \underline{-\,19} \end{array}$$

$$\begin{array}{ccccc} 46 & = & 40+6 & = & 30+16. \\ \underline{19} & \underline{=} & \underline{10+9} & \underline{=} & \underline{10+\ 9.} \end{array}$$

He can then use the missing-addend approach to obtain 30 − 10 = 20, 16 − 9 = 7; so his answer becomes 20 + 7 = 27.

Notice, however, that this computation makes a tacit assumption which is rarely pointed out in elementary texts. The original problem was

$$46 - 19.$$

For convenience, it was first expressed in the form

$$(30 + 16) - (10 + 9).$$

However, the answer was actually computed as

$$(30 - 10) + (16 - 9).$$

It is only fair to ask how we know that this "shifting of terms" yields the correct result. The answer is embodied in the following generalization:

> *For all whole numbers a, b, c, d, where a is not smaller than c, and b is not smaller than d,*
>
> $$(a + b) - (c + d) = (a - c) + (b - d).$$

(We include the following proof for those teachers who are interested in seeing one.)

According to our missing-addend approach to subtraction (see page 72):

If $a - c = x$, *then* $a = c + x$; *and*
if $b - d = y$, *then* $b = d + y$.

---

*A more detailed discussion of subtraction algorithms will appear in a later chapter. At this point we introduce a subtraction algorithm briefly, in order to develop an important property of subtraction.

From these equations, and from the associativity and commutativity of addition, we get

$$\begin{aligned}(a + b) &= (c + x) + (d + y)\\ &= (c + d) + (x + y),\end{aligned}$$

which simply means that

$$(a + b) - (c + d) = x + y.$$

Substituting for x and y, we get

$$(a + b) - (c + d) = (a - c) + (b - d).$$

This is the equality we were trying to establish.

A special case is obtained by letting $d = b$:

$$\begin{aligned}(a + b) - (c + b) &= (a - c) + (b - b)\\ &= (a - c) + 0\\ &= (a - c).\end{aligned}$$

The result is expressed as

$$(a + b) - (c + b) = (a - c).$$

It simply means that the result of a subtraction is unchanged if the same number is added to both numbers in the subtraction problem. For example:

$$\begin{aligned}421 - 97 &= (421 + 3) - (97 + 3)\\ &= 424 - 100\\ &= 324.\end{aligned}$$

DIVISION

6

1. How can division be explained—

  ***a*) Through the use of sets?**

  ***b*) Using the ideas of multiplication?**

2. What is meant by an expression such as "12 ÷ 3"?

Have you ever given your pupils division exercises and told them to check by multiplying? Perhaps one of the exercises was

$$18\overline{)414}$$

If a pupil obtained the answer 23, he was supposed to check by multiplying 23 by 18. He hoped to obtain 414 if his answer to the division exercise was correct.

$$\begin{array}{r} 23 \\ \times\ \underline{18} \\ 184 \\ \underline{230} \\ 414 \end{array}$$

Evidently division has some relationship to multiplication. What is this relationship? What are the consequences of this relationship, and how should they be presented to children?

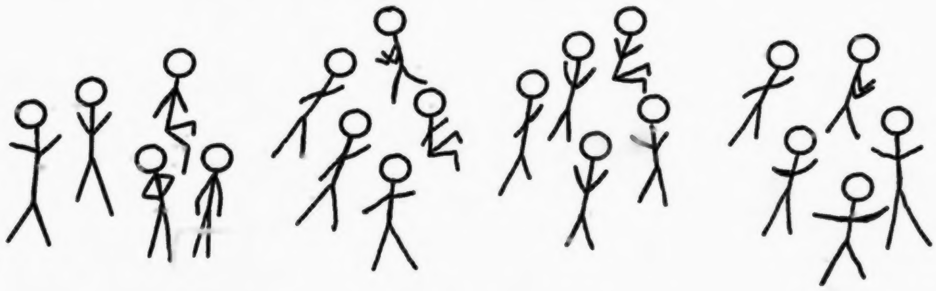
The relationship of division to multiplication rests on the very meaning of division. We teach children that "eight divided by two equals four," but we also need to teach them *why*. If a child says, "Eight divided by two equals six," we must be able to show him why his statement is incorrect. In order to do this, we must go into the very *meaning* of division.

DIVISION

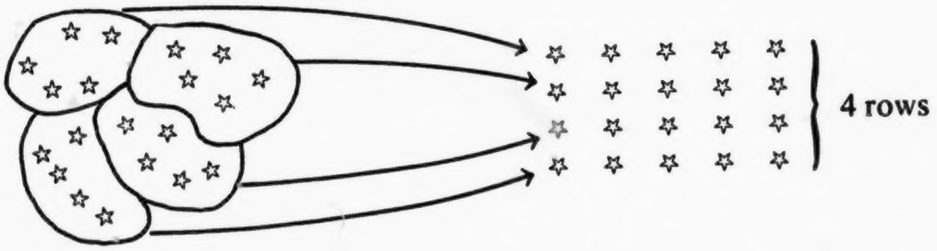
Children usually have little difficulty with problems like this one:

> If 20 boys go to play basketball (5 boys per team), how many teams can be formed?

As indicated in the figure below, no knowledge of arithmetical operations is needed to separate the 20 boys into teams of 5 each.



This separation can also be achieved without the actual presence of the boys. Let a star (☆) stand for each boy. We wish to form teams of 5 boys, so let us arrange the stars in *rows* of five.



The number of rows we can form is the number of teams.

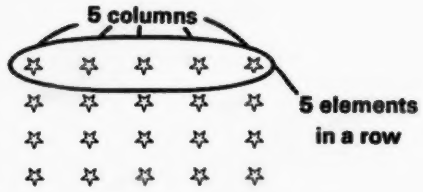
In the language of sets, this problem can be expressed as follows:

> Into how many disjoint sets of 5 elements each can a set of 20 elements be partitioned?

Or, using the language of arrays,* this problem can be expressed as follows:

> If an array has 20 elements, and if each row has 5 elements, how many rows are there?

But in an array the number of elements in a row is the same as the number of columns. So we can again restate the above problem:



> If an array has 20 elements, and if the array has 5 columns, how many rows are there?

---

*The only arrays we shall be concerned with are *rectangular* arrays.

All of the above approaches to the above problem are equivalent. The numerical result is always 4. We may think of any of the above approaches as a way in which the number 4 is obtained from the numbers 20 and 5. To the number pair 20 and 5 division assigns the number 4. We say that 20 divided by 5 is 4. In symbols we write:

$$20 \div 5 = 4.$$

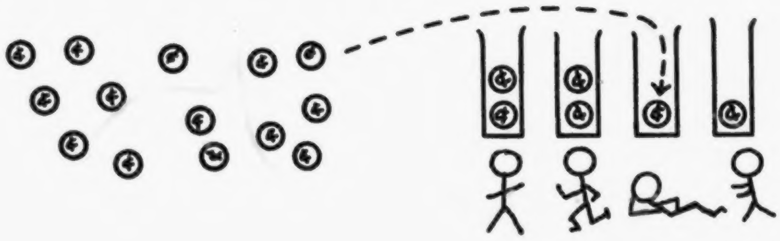
Using arrays, we can define the quotient of a pair of whole numbers as follows:

> *If an array with b elements (where b ≠ 0) has a columns, then the number of rows is named "b ÷ a." We call this the* quotient *of b and a. Division assigns the quotient "b ÷ a" to the pair of whole numbers b and a.*

Problems somewhat different from the one above also fall into the pattern of arrays and are thus an application of division. For example:

> If 20 pennies are being distributed to 4 boys, how many pennies will each boy receive?

If we distribute one penny at a time to each boy, the result is the same as arranging the pennies in an array with one column of pennies assigned to each boy. How many pennies will be in each row?



In such a problem we are given the number of columns and we seek the number of elements in a column. But the number of elements in a column is the same as the number of rows in the array. In other words, we know

(1) the number of elements in an array,
(2) the number of columns,

and we seek

(3) the number of rows.

If we are given the number of elements in an array, then we may be given the number of columns and seek the number of rows or, equivalently, we may be given the number of rows and seek the number of columns. Either case may be considered to be a division problem.

Exercise Set 1

1. Draw arrays to fit these conditions:

- **a.** 12 elements
  2 rows
- **b.** 16 elements
  4 rows
- **c.** 10 elements
  5 rows
- **d.** 6 elements
  1 row

2. How many columns does each array have in Exercise 1?

3. Write two division sentences for each of these arrays:

a.
XXX
XXX
XXX   15 ÷ 5 = 3.
XXX   15 ÷ 3 = 5.
XXX

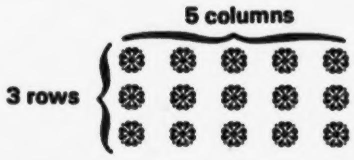
b.
? ? ? ? ? ?
? ? ? ? ? ?
? ? ? ? ? ?
? ? ? ? ? ?

c.
o o
o o
o o
o o
o o
o o
o o
o o

d.
+ + + + + + + + +
+ + + + + + + + +
+ + + + + + + + +
+ + + + + + + + +

In our development of division so far we have not mentioned multiplication. Yet division is often called the "inverse" of multiplication. Why?

Do you recall how arrays are used in explaining multiplication? The product of 3 and 5, for example, is the number of elements in an array with 3 rows and 5 columns.



Therefore, the array has 3 x 5 elements.

In our division examples so far, we were given the number of elements in an array and the number of rows (or columns). So, in effect, we were given a product and one of two factors of that product.

Thus in a division problem, instead of asking, "What is the number of rows of an array with 5 columns and 20 elements?" we could ask, "*5 multiplied by what number will give 20?*"

In other words, to divide 20 by 5, we may complete the sentence $5 \times \square = 20$, or $\square \times 5 = 20$. The sentence $5 \times \square = 20$ has the same meaning as the sentence $20 \div 5 = \square$.

Numbers that are multiplied to form a product are called *factors* (of that product). Thus, the problem of completing the sentence $5 \times \square = 20$ may be considered as "finding the missing factor." So division is concerned with finding the missing factor. Whereas in multiplication we seek a product of two given factors, in division we seek one of the factors of a given product. This is why division is often called the inverse of multiplication.

Approaching division through multiplication becomes increasingly important in later grades, when pupils have to work with fractional numbers and negative numbers. As pupils advance, they should be brought to understand that $20 \div 5$ is that number which, when multiplied by 5, gives 20. Of course, the interpretation of division in terms of sets and arrays is still useful.

We may formally state the missing-factor approach to division as follows:

> *Division assigns to the pair of whole numbers a and b the missing factor in the sentence* $b \times \square = a$, *provided there is exactly one whole number that fits the sentence.*

The missing factor is named "$a \div b$." It is also called the *quotient* of a and b.

A pupil who is introduced to division by finding missing factors need not memorize "division facts." For example, to complete the sentence

$$32 \div 4 = \square,$$

a child may think

$$32 = \square \times 4$$

or

$$32 = 4 \times \square$$

and complete the sentence from his knowledge of multiplication facts. This will involve trial and error at first.

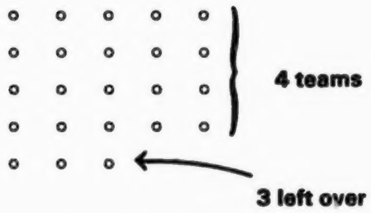
A problem closely related to division comes up in connection with situations similar to that of the 20 boys playing in teams of 5. What if

23 boys want to play basketball? Can we *divide* 23 by 5? If we could, we should find the result by completing

$$5 \times \square = 23.$$

But no whole number exists to complete this sentence correctly, so there is no whole-number meaning for $23 \div 5$. When working with just the whole numbers, we would say that 23 is not divisible by 5.

In later grades children will learn that fractional numbers can be used to complete correctly sentences such as $5 \times \square = 23$. But the concrete problem remains: How can we decide, mathematically, how many teams of 5 can be formed from 23 boys? The answer may be obtained by attempting to complete an array:



This shows that 4 teams can be formed, but there will be 3 players "left over."

Another approach is the following:

> *For the whole numbers 23 and 5, we determine whole numbers to complete the sentence* $23 = (5 \times \square) + \triangle$ *in such a way that the number for the* $\triangle$ *is as small as possible (in this case less than 5). These numbers are 4 and 3, since* $23 = (5 \times 4) + 3$.

Once again we see that 4 teams of 5 can be formed, with 3 boys remaining.

In general, if we are given a set of b elements, and if we want to form disjoint subsets of a elements each, we then seek to complete the sentence

$$b = (a \times \square) + \triangle,$$

where the number for the $\triangle$ (called *the remainder*) is less than a.

Division problems with a remainder are treated further in Chapter 9, "Division Algorithms."

Exercise Set 2

1. Write two division sentences related to each of these multiplication sentences.

a. $6 \times 7 = 42.$ $6 = 42 \div 7.$ $7 = 42 \div 6.$

b. $3 \times 1 = 3.$ ________ ________

c. $9 \times 10 = 90.$ ________ ________

d. $10 \times 13 = 130.$ ________ ________

e. $4 \times 8 = 32.$ ________ ________

2. Which of these sentences can be completed by a whole-number missing factor?

a. $6 \times \square = 30.$

b. $6 \times \square = 35.$

c. $\square \times 9 = 99.$

d. $0 \times \square = 10.$

e. $2 \times \square = 2.$

f. $3 \times \square = 0.$

g. $6 \times \square = 64.$

h. $\square \times 12 = 13.$

i. $\square \times 15 = 0.$

j. $4 \times \square = 152.$

3. For each division sentence write a related multiplication sentence, then complete both sentences.

a. $8 \div 2 = \boxed{4}.$ $8 = \boxed{4} \times 2$ (or $8 = 2 \times \boxed{4}$).

b. $6 \div 6 = \square.$ ________

c. $12 \div 1 = \square.$ ________

d. $0 \div 8 = \square.$ ________

e. $55 \div 11 = \square.$ ________

4. Which whole numbers will complete this sentence?

$$0 \times \square = 0.$$

5. Complete these sentences with whole numbers. In each case use the smallest possible whole number for the $\triangle$. In each sentence the number used for the $\triangle$ should be less than the given factor.

a. $62 = (7 \times \boxed{8}) + \overset{6}{\triangle}.$

b. $6 = (4 \times \square) + \triangle.$

c. $5 = (7 \times \square) + \triangle.$

d. $55 = (11 \times \square) + \triangle$.

e. $57 = (7 \times \square) + \triangle$.

PROPERTIES OF DIVISION

In the new mathematics programs, children learn not only the meaning of addition and multiplication but also the properties of these mathematical operations. Two important properties of addition and of multiplication are the associative property and the commutative property.

Associative and Commutative Properties
of Multiplication and Addition

Operation	Associative Property	Commutative Property
Multiplication	For all whole numbers a, b, and c, $(a \times b) \times c = a \times (b \times c)$. *Example:* $(3 \times 6) \times 4 = 3 \times (6 \times 4)$.	For all whole numbers a and b, $a \times b = b \times a$. *Example:* $12 \times 7 = 7 \times 12$.
Addition.	For all whole numbers a, b, and c, $(a + b) + c = a + (b + c)$. *Example:* $(3 + 6) + 4 = 3 + (6 + 4)$.	For all whole numbers a and b, $a + b = b + a$. *Example:* $12 + 7 = 7 + 12$.

Does division have these properties also? Let us consider two examples:

1. *Is division commutative*? For example, does $12 \div 4 = 4 \div 12$? Clearly $12 \div 4 = 3$ because 3 completes the sentence "$12 = \square \times 4$." But "$4 \div 12$" is not a name for 3; in fact it is not a name for any whole number, because no whole number fits the sentence $12 \times \square = 4$. So $12 \div 4 \neq 4 \div 12$. (Using rational numbers, we would find that $12 \div 4 = 3$ and $4 \div 12 = \frac{1}{3}$; and yet, here too, $4 \div 12 \neq 12 \div 4$.)

This exception (although there are many others) is sufficient to show that *division is not commutative*. (For division to be commutative, it would be necessary that $a \div b = b \div a$ for *all* whole numbers a and b.)

2. *Is division associative*?
For example, does $16 \div (8 \div 2) = (16 \div 8) \div 2$?

$$16 \div (8 \div 2) = 16 \div 4 = 4,$$

but

$$(16 \div 8) \div 2 = 2 \div 2 = 1.$$

So, $16 \div (8 \div 2) \neq (16 \div 8) \div 2$, showing that *division is not associative*. (For division to be associative it would be necessary that

$$(a \div b) \div c = a \div (b \div c)$$

for all whole numbers a, b, and c.)

Notice, however, that there are special cases which could confuse children. For example, does $(16 \div 8) \div 1$ equal $16 \div (8 \div 1)$? Yes. A child might say, "Division is *sometimes* associative." However, we may apply the terms associativity or commutativity only when these properties hold in *all* cases. A single exception is sufficient to show that an operation *is not* commutative (or associative), but specific examples can never show that an operation *is* commutative (or associative).

We have come to the same conclusions about division as we did about subtraction: neither is commutative, neither is associative.

Exercise Set 3

1. Insert "=" or "≠," whichever applies, in each circle.

   a. $6 \div 2$ ○ $2 \div 6$.

   b. $(6 \div 2) \div 1$ ○ $6 \div (2 \div 1)$.

   c. $(16 \div 4) \div 2$ ○ $16 \div (4 \div 2)$.

   d. $(12 \div 6) \times 2$ ○ $12 \div (6 \times 2)$.

   e. $12 \times (6 \div 2)$ ○ $(12 \times 6) \div 2$.

2. Insert parentheses to make each sentence true.

   a. $8 \div 4 \times 2 = 1$.

   b. $8 \div 4 \div 2 = 4$.

   c. $12 \div 3 + 1 = 3$.

   d. $12 \div 3 - 1 = 3$.

   e. $12 \div 3 \times 2 = 2$.

ZERO AND ONE IN DIVISION

We have already noted that the numbers 0 and 1 play special roles in multiplication.

MULTIPLICATION PROPERTY OF 1	MULTIPLICATION PROPERTY OF 0
For every whole number a, $a \times 1 = a$ and $1 \times a = a$. *Examples:* $5 \times 1 = 5$. $1 \times 16 = 16$.	For every whole number a, $a \times 0 = 0$ and $0 \times a = 0$. *Examples:* $5 \times 0 = 0$. $0 \times 16 = 0$.

These special properties lead to some interesting facts about 0 and 1 in division. They are probably best communicated to children through examples and exercises.

ONE IN DIVISION

The multiplication property of 1 leads to such sentences as $5 \times 1 = 5$, $1 \times 16 = 16$, $1 \times 2 = 2$, $65 \times 1 = 65$, etc. From each of these multiplication sentences, two division sentences may be derived.

$5 \times 1 = 5$ leads to $5 \div 5 = 1$ and $5 \div 1 = 5$.
$1 \times 16 = 16$ leads to $16 \div 16 = 1$ and $16 \div 1 = 16$.
$1 \times 2 = 2$ leads to $2 \div 2 = 1$ and $2 \div 1 = 2$.
$65 \times 1 = 65$ leads to $65 \div 65 = 1$ and $65 \div 1 = 65$.

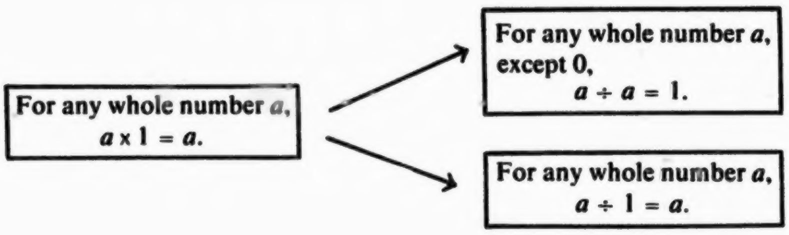
These results suggest two generalizations.

1. The fact that $5 \div 5 = 1$, $16 \div 16 = 1$, etc., suggests that *any whole number divided by itself yields 1.*

Actually, we shall see in the next section that there is *one* exception. We do not have $0 \div 0 = 1$. In fact we shall see that the expression "$0 \div 0$," and indeed any expression "$a \div 0$," is meaningless.

2. The fact that $5 \div 1 = 5$, $16 \div 1 = 16$, etc., suggests that *any whole number divided by 1 yields the given whole number.*

These generalizations govern the role of 1 in division. They can be derived from the multiplication property of 1.


For any whole number a, a × 1 = a.
For any whole number a, except 0, a ÷ a = 1.
For any whole number a, a ÷ 1 = a.

ZERO IN DIVISION

The number 0 can appear in division exercises in three ways:

(1) 0 divided by some other number

(2) Some other number divided by 0
(3) 0 divided by 0

We shall investigate these three possibilities, relying always on the multiplication properties of 0:

For every whole number a, $a \times 0 = 0$ and $0 \times a = 0$.

If neither a nor b is 0, then $a \times b$ cannot be 0.

(1) *0 divided by some other number*

The problem here is to decide what numbers, if any, are named by expressions such as "$0 \div 4$," "$0 \div 12$," "$0 \div 1$," "$0 \div 75$."

Earlier in this chapter we found what number is named by an expression such as $20 \div 5$. To do that, we relied on the missing-factor approach to division. Thus, $20 \div 5$ is that number (there is only one such number) which, when multiplied by 5, gives 20.

$$5 \times \square = 20.$$

$$\text{Since } 5 \times 4 = 20, \text{ then } 20 \div 5 = 4.$$

(Note that 5, multiplied by any number other than 4, cannot yield 20 as the product.)

In the same way, $0 \div 4$ is that number (there is only one such number) which, when multiplied by 4, gives 0.

$$4 \times \square = 0.$$

$$\text{Since } 4 \times 0 = 0, \text{ we have found that } 0 \div 4 = 0.$$

(Note that 4, multiplied by any number other than 0, cannot yield 0 as the product.)

Similarly, by completing

$$12 \times \square = 0, \qquad 1 \times \square = 0, \qquad 25 \times \square = 0,$$

we find that

$$0 \div 12 = 0, \qquad 0 \div 1 = 0, \qquad 0 \div 25 = 0.$$

(Notice that we have avoided $0 \div 0$; this special case is handled later.)

These examples suggest that "0 divided by any whole number (except 0) yields 0." More formally stated:

For any whole number a, if $a = 0$ *then* $0 \div a = 0$.

This generalization arises from the fact that there is one and only one missing factor in any sentence of the form $a \times \square = 0$ (where $a \neq 0$). This missing factor is the number 0 itself.

(2) *Some number, other than 0, divided by 0*

The problem here is to decide what numbers, if any, are named by expressions such as "$4 \div 0$," "$8 \div 0$," "$1 \div 0$," "$24 \div 0$."

Again, we use the missing-factor definition of division to investigate the problem. If 8 ÷ 0 is a number, it must be a number which, when multiplied by 0, yields 8.

$$0 \times \square = 8.$$

But no whole number can make this true, since 0 times any whole number is 0. So "8 ÷ 0" does not name a whole number; the expression "8 ÷ 0" is meaningless.

Analyzing 4 ÷ 0, 1 ÷ 0, and 24 ÷ 0 leads to the sentences

$$0 \times \square = 4, \quad 0 \times \square = 1, \quad 0 \times \square = 24.$$

No numbers can make these sentences true, because the product of 0 and any number must be 0. Thus, the expressions "8 ÷ 0," "4 ÷ 0," "1 ÷ 0," "24 ÷ 0," etc., are meaningless. These examples suggest that *division by zero is meaningless*. Before we can assert this generalization, we must investigate the possibility of dividing 0 by 0.

(3) *0 divided by 0*

The problem here is to decide what number, if any, is named by the expression "0 ÷ 0."

Using the missing-factor definition of division, 0 ÷ 0 would be a number which, when multiplied by 0, yields 0.

$$0 \times \square = 0.$$

Are there any such numbers? Unfortunately *every* whole number will fit this sentence.

$$0 \times \boxed{0} = 0, \quad 0 \times \boxed{1} = 0, \quad 0 \times \boxed{2} = 0, \quad \text{etc.}$$

These equalities suggest that 0 ÷ 0 is 1, 2, 3, ... However, "0 ÷ 0" can't be the name of every whole number because, for example, if we had 0 ÷ 0 = 4 and 0 ÷ 0 = 6, we would obtain the inconsistency 4 = 6. So "0 ÷ 0" is not the name of a unique number, and therefore "0 ÷ 0" is given no meaning at all.

We may now assert a general fact of arithmetic:

Division by 0 is meaningless.

We might remark here that division by 0 will remain meaningless even when other numbers, such as the negative and rational numbers, are considered.

Exercise Set 4

1. If an array has 9 rows and 9 elements altogether, how many columns does it have? What division sentence expresses this result?

2. Which of these expressions name(s) the number 0?

a. $0 \div 7$
b. $14 - 14$
c. $9 - 0$
d. 1×0
e. $0 \div 0$
f. $0 \div 1$

3. Which of these expressions name(s) a whole number?

a. $6 \div 4$
b. $8 \div 16$
c. $0 \div a$
d. $6 \div 0$
e. $0 \div 0$
f. $13 \div 3$

SUMMARY

Subtraction and division are similar because they are similarly related to addition and multiplication, respectively. Consequently, they have similar properties. Because 0 plays a special role in multiplication, 0 plays a special role in division.

To summarize: What does "$8 \div 2$" mean? $8 \div 2$ is the unique number that fits the sentence

$$2 \times \square = 8.$$

If the product of two factors is divided by one of these factors, the quotient is the remaining factor.

Division is neither commutative nor associative.

Any number (except 0) divided by itself yields the quotient 1.

Any number divided by 1 yields that number.

0 divided by any number (except 0) yields 0.

Division by 0 has no meaning.

7

ADDITION AND SUBTRACTION ALGORITHMS

1. **What is meant by *compute*?**
2. **What is meant by *algorithm*?**
3. **How do we justify the traditional addition algorithm?**
4. **How do we justify the traditional subtraction algorithm?**

OPERATIONS AND COMPUTATION

We have seen that addition assigns to any pair of numbers, a and b, a number, $a + b$, called their sum. A definition was given in terms of sets:

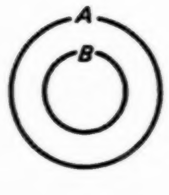
> *For any two whole numbers, a and b, let A and B be disjoint sets such that* $n(A) = a$ *and* $n(B) = b$. *Then*
> $$a + b = n(A \cup B).$$

By using this definition, it is possible to derive the fundamental properties of addition:

1. $a + b = b + a$ (commutative property)
2. $(a + b) + c = a + (b + c)$ (associative property)
3. $a + 0 = a$ (addition property of 0)

Subtraction can also be defined in terms of sets:

> *Suppose the whole number a is greater than, or the same as, the whole number b. Now let A be a set such that* $n(A) = a$, *and let B be a subset of A such that* $n(B) = b$. *Then* $a - b$ *is the number of elements in A which are not in B.*



We have seen that this definition is equivalent to defining subtraction in terms of addition. If a and b are whole numbers, with a greater than or the same as b, then there is a whole number c such that $a = b + c$. Furthermore, if A and B are the sets described above, c is the number of elements in set A which are not in set B. Thus, we can define $a - b$ to be the number c such that $a = b + c$.

This chapter will present computational techniques related to addition and subtraction. However, before going into the techniques, let's be certain that we understand the nature of computation. The sum of 3 and 5 may be denoted by

$$3 + 5.$$

But we memorize a standard name or symbol for this number, namely

$$8,$$

and we write

$$3 + 5 = 8.$$

The sum of 432 and 359 may be denoted by

$$432 + 359.$$

We know that there is a standard name for this sum, too. However, we hardly care to memorize it. Instead, by making use of (1) the design of the numeration system, (2) the combinations that we have memorized, and (3) the properties of the addition operation, we can derive schemes called *algorithms* (or *algorisms*) for readily determining such standard names. The determination of standard names is the essence of computation. We begin with a certain name for a number and proceed to the standard name. Because of this, some people refer to computation as a name-changing process.

ADDITION COMPUTATION

Our numeration system has two main features—it is a place-value, or positional, system; and its base is ten. The first of these features is the more important as far as computation is concerned. We might say that it is this aspect of the numeration system that allows us to "do what we do" in computation.

Let us recall that the numeral 457, for example, in the base-ten numeration system means

$$(4 \times 10^2) + (5 \times 10) + (7 \times 1).$$

Actually "457" can be thought of as an abbreviation for this "expanded form." Of course, children who have not studied multiplication and exponents are not yet ready for the above form, but they can write 457 as

$$4 \text{ hundreds} + 5 \text{ tens} + 7 \text{ ones}$$

or as

$$400 + 50 + 7.$$

Also, given either of the latter forms, they should recognize it as a name for the number 457.

As a first illustration of what is involved in computation, consider the sum

$$15 + 8.$$

Because of the way our numeration system is set up, we know that $15 = 10 + 5$. Hence, we can write

$$15 + 8 = (10 + 5) + 8.$$

By applying the associative property, we have

$$(10 + 5) + 8 = 10 + (5 + 8).$$

Now, children usually learn that $5 + 8$ is the same number as 13 and that 13 is the same number as $10 + 3$. So we have

$$10 + (5 + 8) = 10 + (10 + 3);$$

and another application of the associative property gives

$$10 + (10 + 3) = (10 + 10) + 3.$$

Finally, since $10 + 10 = 20$, we arrive at

$$(10 + 10) + 3 = 20 + 3 = 23.$$

Summarizing:

$$\begin{aligned} 15 + 8 &= (10 + 5) + 8 \\ &= 10 + (5 + 8) \\ &= 10 + 13 \\ &= 10 + (10 + 3) \\ &= (10 + 10) + 3 \\ &= 20 + 3 \\ &= 23. \end{aligned}$$

In computing a sum such as

$$43 + 25,$$

we first use our knowledge of the decimal numeration system to rename 43 and 25 thus:

$$43 + 25 = (40 + 3) + (20 + 5).$$

We know that the associative and commutative properties allow us to rearrange the addends:

$$(40 + 3) + (20 + 5) = (40 + 20) + (3 + 5).$$

Since $40 + 20 = 60$ and $3 + 5 = 8$, the sum is $60 + 8$, or 68. We can display our work as follows:

$$\begin{aligned} 43 + 25 &= (40 + 3) + (20 + 5) \\ &= (40 + 20) + (3 + 5) \\ &= 60 + 8 \\ &= 68. \end{aligned}$$

This is but one of the systematic procedures or algorithms for computing a sum.

The above explanation should raise one important question: From what has been said, how do we know that $40 + 20 = 60$? A complete explanation of this can rest on the distributive property (which has not yet been discussed) and on the facts that 40 means 4×10 and 20 means 2×10. But when this question first arises in the elementary school, multiplication need not be involved. At this stage the children interpret 40 as 4 tens (that is, the number for 4 "bundles," each bundle consisting of ten objects.) Since 40 is 4 tens and 20 is 2 tens, $40 + 20$ is 6 tens or 60. Some time should be spent on sums of this sort before going into examples of the type being discussed above. Likewise, before considering sums such as $327 + 253$ it is necessary to discuss sums of multiples of 100. We'll assume in what follows that the necessary work with multiples of 10 and 100 has been done.

Let us return to the sum $43 + 25$. Once we realize that we can rearrange addends as we please because addition is both commutative and associative, it is easy to devise other ways of displaying our work. The form shown below is slightly more compact:

$$\begin{array}{r} 43 = 40 + 3. \\ \underline{25 = 20 + 5.} \\ 60 + 8 = 68. \end{array}$$

Bear in mind, however, that although it is not apparent that commutativity and associativity have been used when the work is shown this way, justification of this form rests on these two properties.

It is, of course, possible to shorten the work further. We might still *think* exactly as above, but simply write:

$$\begin{array}{r} 43 \\ \underline{25} \\ 60 \\ \underline{8} \\ 68 \end{array} \qquad \text{or} \qquad \begin{array}{r} 43 \\ \underline{25} \\ 8 \\ \underline{60} \\ 68 \end{array}$$

Finally, we might show the work in the extremely compact form familiar to all:

$$\begin{array}{r} 43 \\ \underline{25} \\ 68 \end{array}$$

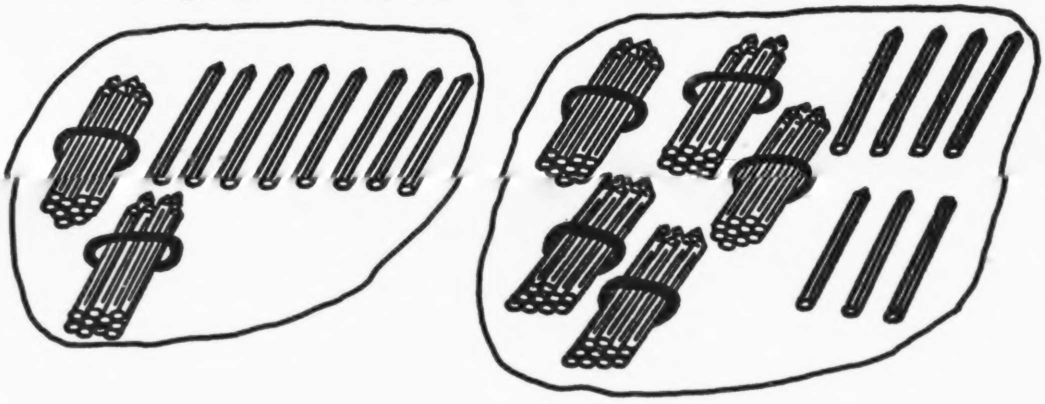
It is important that we do not move to the compact form too quickly in teaching children. The process tends to become purely mechanical for far too many elementary school children. If one of the longer forms is used, a child is compelled to think about what is being written. The short form should be introduced only when we feel that the child thoroughly understands the important ideas that underlie the addition computation. Even when he finally uses the short form, he should remain aware of the basic ideas on which the process rests.

As a further illustration of what is involved in computation, let us consider the sum

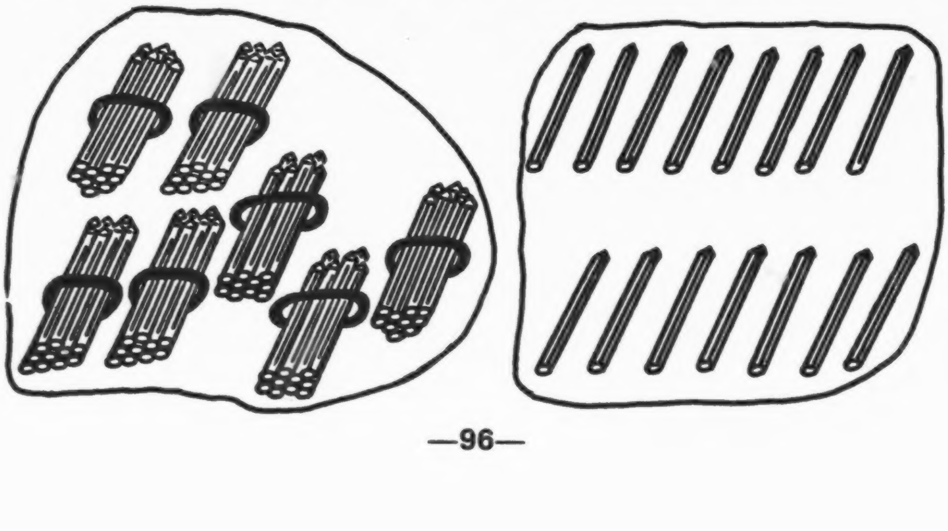
$$28 + 57.$$

In analyzing this for a child, we might begin with two sets of physical objects.

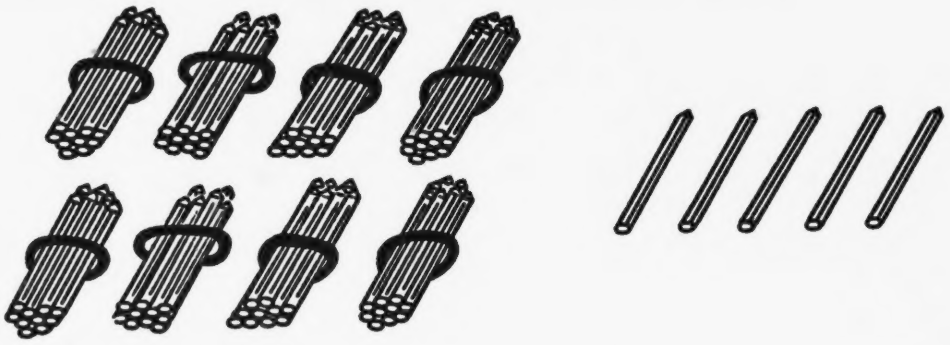
We have (2 tens + 8 ones) + (5 tens + 7 ones).



Regrouping gives us (2 tens + 5 tens) + (8 ones + 7 ones).



We realize that the 15 ones can be grouped to produce another ten and still leave five ones. The ten is put with the other 7 tens.



Since we now have 8 tens + 5 ones, we say that the computed sum is 85.

Displaying this reasoning in a fashion similar to the earlier example, we have:

$$\begin{aligned}
28 + 57 &= (20 + 8) + (50 + 7)\\
&= (20 + 50) + (8 + 7)\\
&= 70 + 15\\
&= 70 + (10 + 5)\\
&= (70 + 10) + 5\\
&= 80 + 5\\
&= 85.
\end{aligned}$$

You should recognize that so-called "carrying" is nothing more than regrouping—that is, an application of the associative property. This takes place in moving from the fourth line to the fifth in the above display.

Here again, the work can be displayed more briefly:

$$\begin{array}{l}
\quad 28 = 20 + 8\\
\quad \underline{57 = 50 + 7}\\
\qquad\quad 70 + 15\\
70 \leftarrow\!\!\lrcorner \qquad \big|\\
\underline{10 + 5} \leftarrow\!\!\lrcorner\\
\overline{80 + 5} = 85
\end{array}$$

After we are sure that the children understand the underlying concepts, we can suggest that they use either of the following forms:

$$\begin{array}{rcr}
28 & & 28\\
\underline{57} & \swarrow 20 + 50 \searrow & \underline{57}\\
70 & & 15\\
\underline{15} & \leftarrow 8 + 7 \nearrow & \underline{70}\\
85 & & 85
\end{array}$$

Eventually, we encourage a child to think, "8 + 7 is 15. I'll write down the 5 and remember 10. Then 10 + 20 + 50 is 80." All that would appear on the paper is

$$\begin{array}{r} 28 \\ \underline{57} \\ 85 \end{array}$$

Let's consider one more example of an addition computation. Here are several different ways of displaying the work in computing 283 + 54 + 105. Remember that no matter how much or how little we actually write, the reasoning is essentially the same.

1.

$$\begin{aligned} 283 + 54 + 105 &= (200 + 80 + 3) + (50 + 4) + (100 + 5) \\ &= (200 + 100) + (80 + 50) + (3 + 4 + 5) \\ &= 300 + 130 + 12 \\ &= 300 + (100 + 30) + (10 + 2) \\ &= (300 + 100) + (30 + 10) + 2 \\ &= 400 + 40 + 2 \\ &= 442. \end{aligned}$$

2.

$$\begin{array}{rl} 283 = & 200 + 80 + 3 \\ 54 = & \phantom{200 + {}} 50 + 4 \\ \underline{105 =} & \underline{100 \phantom{{} + 80} + 5} \\ & 300 + 130 + 12 = 300 + (100 + 30) + (10 + 2) \\ & = (300 + 100) + (30 + 10) + 2 \\ & = 400 + 40 + 2 \\ & = 442. \end{array}$$

$$\begin{array}{l} 300 \leftarrow 300 \\ 100 + 30 \leftarrow 130 \\ \underline{\phantom{100 + {}}10 + 2} \leftarrow 12 \\ 400 + 40 + 2 = 442. \end{array}$$

3.

$$\begin{array}{r} 283 \\ 54 \\ \underline{105} \\ 12 \\ 130 \\ \underline{300} \\ 442 \end{array}$$

4.

$$\begin{array}{r} 283 \\ 54 \\ \underline{105} \\ 442 \end{array}$$

The concepts at the heart of computation are brought out best by the first form shown. You will note that, as in the preceding examples, we (1) made use of what we know about our numeration system, (2) relied on previously memorized facts, and (3) used the properties of addition. Once again, it is important that children *understand* these underlying ideas before using a shorter form for computation.

Exercise Set 1

1. Below is shown the computation of 53 + 15 and 27 + 35. Fill in the frames correctly:

a.

$$\begin{aligned} 53 + 15 &= (50 + 3) + (10 + \square) \\ &= (50 + \triangle) + (3 + 5) \\ &= 60 + \Diamond \\ &= 68. \end{aligned}$$

b.

$$\begin{aligned} 27 + 35 &= (\square + 7) + (30 + \triangle) \\ &= (\square + 30) + (7 + \triangle) \\ &= 50 + 12 \\ &= 50 + (\Diamond + 2) \\ &= (50 + \Diamond) + 2 \\ &= 60 + 2 \\ &= 62. \end{aligned}$$

2. Compute each of the following sums by using a form like that displayed in the above exercise.

a. 19 + 67

b. 173 + 8

c. 231 + 36

d. 97 + 24

e. 208 + 523

f. 145 + 278

SUBTRACTION COMPUTATION

We shall now see that the same general ideas apply to subtraction. Subtraction has been defined in such a way that if we know addition "facts," then we know subtraction "facts." For example, the number that completes

$$7 - 2 = \square$$

so as to form a true sentence is, by definition, the same as the number that completes

$$7 = \square + 2.$$

Thus, since $7 = 5 + 2$, we can write $7 - 2 = 5$. Similarly, we can write $15 - 8 = 7$ because $15 = 7 + 8$.

Determining the standard name for $39 - 15$ is almost as easy. We know that 39 means 3 tens + 9 ones, while 15 means 1 ten + 5 ones. We also know that $39 - 15$ is that number which, added to 15, gives the sum 39. So we wish to complete the following:

$$(1 \text{ ten} + 5 \text{ ones}) + (\triangle \text{tens} + \Diamond \text{ ones}) = 3 \text{ tens} + 9 \text{ ones}.$$

By using the commutative and associative properties of addition, we can rearrange the terms as follows:

$$(1 \text{ ten} + \triangle \text{ tens}) + (5 \text{ ones} + \Diamond \text{ ones}) = 3 \text{ tens} + 9 \text{ ones}.$$

The easiest way to find correct numbers for the frames is to complete the two sentences

$$1 \text{ ten} + \triangle \text{ tens} = 3 \text{ tens}, \quad \text{and} \quad 5 \text{ ones} + \Diamond \text{ ones} = 9 \text{ ones}.$$

Using 2 for the $\triangle$ correctly completes the first sentence, while inserting 4 for the $\Diamond$ correctly completes the second sentence. Hence $39 - 15$ is 2 tens + 4 ones, or 24. The work can be arranged in a fashion similar to that for addition:

$$\begin{array}{rl} 39 = & 3 \text{ tens} + 9 \text{ ones} \\ - \underline{15 =} & \underline{1 \text{ ten} \;\; + 5 \text{ ones}} \\ & 2 \text{ tens} + 4 \text{ ones} = 24 \end{array}$$

or

$$\begin{array}{rl} 39 = & 30 + 9 \\ - \underline{15 =} & \underline{10 + 5} \\ & 20 + 4 \; = 24 \end{array}$$

Now consider this number sentence:

$$63 - 28 = \square.$$

This is equivalent to

$$28 + \square = 63.$$

Or we could write

$$(2 \text{ tens} + 8 \text{ ones}) + (\triangle \text{tens} + \Diamond \text{ ones}) = 6 \text{ tens} + 3 \text{ ones}.$$

Now if we follow the same procedure as in the preceding example, we arrive at the two sentences

$$2 \text{ tens} + \triangle \text{tens} = 6 \text{ tens}, \quad \text{and} \quad 8 \text{ ones} + \Diamond \text{ ones} = 3 \text{ ones}.$$

Unfortunately, the latter sentence cannot be completed with a whole number. However, once more the associative property comes to our aid. Since

$$\begin{aligned}\text{6 tens} + \text{3 ones} &= (\text{5 tens} + \text{1 ten}) + \text{3 ones}\\ &= \text{5 tens} + (\text{1 ten} + \text{3 ones})\\ &= \text{5 tens} + \text{13 ones},\end{aligned}$$

we have

$$(\text{2 tens} + \text{8 ones}) + (\triangle\ \text{tens} + \Diamond\ \text{ones}) = \text{5 tens} + \text{13 ones}.$$

We then have

$$(\text{2 tens} + \triangle\ \text{tens}) + (\text{8 ones} + \Diamond\ \text{ones}) = \text{5 tens} + \text{13 ones},$$

from which we derive the two sentences

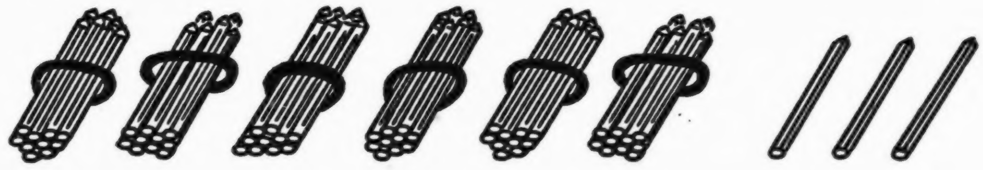
$$\text{2 tens} + \triangle\ \text{tens} = \text{5 tens}, \qquad \text{and} \qquad \text{8 ones} + \Diamond\ \text{ones} = \text{13 ones}.$$

We complete these sentences with 3 for the $\triangle$ and 5 for the $\Diamond$, and conclude that 63 − 28 is 3 tens + 5 ones, or 35. As before, this can be displayed in different ways:

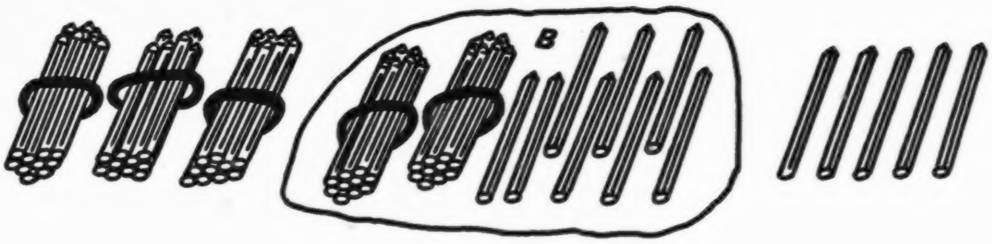
$$\begin{array}{rl} 63 = \text{6 tens} + \text{3 ones} = & \text{5 tens} + \text{13 ones}.\\ \underline{-28 = \text{2 tens} + \text{8 ones} =} & \underline{\text{2 tens} + \ \ \text{8 ones}.}\\ & \text{3 tens} + \ \ \text{5 ones} \ = 35.\end{array}$$

$$\begin{array}{rl} 63 = 60 + 3 = & 50 + 13.\\ \underline{-28 = 20 + 8 =} & \underline{20 + \ \ 8.}\\ & 30 + \ \ 5 \ = 35.\end{array}$$

Even if a child continues to think of subtraction in terms of a set and one of its subsets, he will arrive at essentially the same process. In the example above, he would begin with a set (call it *A*) of 63 objects:



After regrouping, it is easy to identify a subset (call it *B*) of 28 objects:



Then the set of objects in *A* but not in *B* consists of 3 tens + 5 ones, or 35 objects. Notice that if a child thinks of subtraction in this way, he can still display his work in either of the forms above.

Below is one more illustration of a subtraction computation:

$$423 - 157$$

$$\begin{array}{r}
423 = 400 + 20 + 3 = 400 + 10 + 13 = 300 + 110 + 13. \\
-157 = 100 + 50 + 7 = 100 + 50 + 7 = 100 + 50 + 7. \\
\hline
200 + 60 + 6 = 266.
\end{array}$$

The child who displays the computation in this form is much less likely to lose sight of the ideas involved. Of course, he should eventually get to the point where he writes only:

$$\begin{array}{rrr}
{}^{3} & {}^{11} & {}^{1} \\
\cancel{4} & \cancel{2} & 3 \\
-1 & 5 & 7 \\
\hline
2 & 6 & 6
\end{array}$$

But this should come *after* considerable work on the basic concepts, and even then he should be able to explain his work if called upon to do so.

Exercise Set 2

Use expanded notation to compute the following differences.

1. 78 – 23
2. 63 – 7
3. 52 – 39
4. 348 – 92
5. 403 – 126
6. 500 – 278

The subtraction algorithms discussed above are perfectly general. They are applicable to all subtraction problems concerned with whole numbers. There are shortcuts, however, that one who is observant can often use. Many children are capable of learning the underlying ideas for such shortcuts and will be on the lookout for a chance to apply them.

We shall discuss one useful idea here. To introduce this idea, let us consider two persons whose ages are 17 and 12. The difference between their ages is 5. What will be the difference in their ages 14 years from now? 25 years from now? Of course the difference will still be 5, no matter how many years from now we want to consider. This, then, is saying that

$$17 - 12 = (17 + 14) - (12 + 14)$$

or

$$17 - 12 = (17 + 25) - (12 + 25)$$

or in general

$$17 - 12 = (17 + c) - (12 + c),$$

where c is any number.

In fact, we could say that if a is a number greater than or the same as a number b, then

$$a - b = (a + c) - (b + c)$$

for any number c. We can express this idea as follows:

> *When each of two numbers is increased by the same amount, the difference between the resulting numbers is the same as the difference between the original numbers.*

How can this idea be used in subtraction computation? The usual algorithm for computing the difference 427 − 299 is somewhat complicated. However, by using the principle just discussed, it is easy. We choose c to be 1, and we have

$$427 - 299 = (427 + 1) - (299 + 1) = 428 - 300 = 128.$$

It should be clear why c was chosen to be 1.

Exercise Set 3

1. Show an easy way to compute

a. 629 − 297

b. 4,384 − 1,995

2. Some people have learned a different subtraction algorithm. To compute 53 − 26, they would add 10 to the 3 in 53 and also add 10 to the 2 tens in 26. So their work would look like this:

$$\begin{array}{r} 50 + 13 \\ 30 + \ \ 6 \\ \hline 20 + \ \ 7 \end{array} = 27 \qquad \text{or} \qquad \begin{array}{r} 5\ {}^{1}3 \\ 2_{1}\ 6 \\ \hline 2\ \ 7 \end{array}$$

Show that this is just another way to apply the principle expressed by

$$a - b = (a + c) - (b + c).$$

COMPUTATION IN ANOTHER BASE

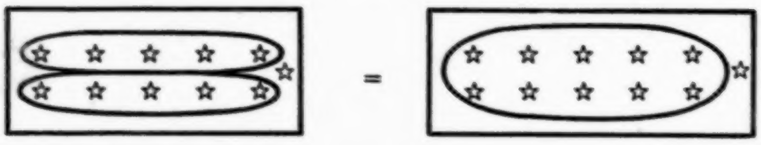
Earlier it was stated that the positional feature of our numeration system is more important for computation than the fact that the base is ten. To illustrate this, let us now examine addition computation in a base-five system.

In the base-five numeration system, the numeral "21_{five}" is read "two one, base five" and means 2 fives + 1 one. This, of course, is the number eleven. Thus

$$21_{\text{five}} = 11_{\text{ten}}.^{*}$$

The different numerals that appear in the above equation name the same number. They suggest different ways of grouping:

$$2 \text{ fives} + 1 \text{ one} = 1 \text{ ten} + 1 \text{ one}.$$



Similarly

$$13_{\text{five}} \text{ means } 1 \text{ five} + 3 \text{ ones}.$$

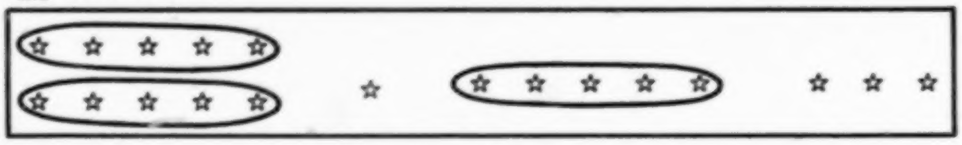
So

$$13_{\text{five}} = 8_{\text{ten}}.$$

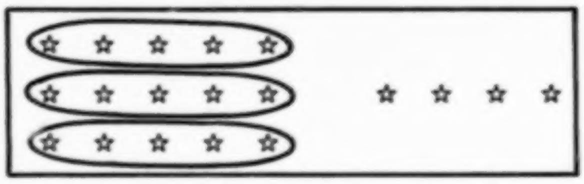
Now, proceeding to computation, we could ask: What is the standard numeral in base five for

$$21_{\text{five}} + 13_{\text{five}}?$$

We still mean the same thing by the plus sign, so the above sum is the number of elements in the union of a set of 21_{five} elements with a set of 13_{five} elements:



We see that the number of elements in the union is 3 fives + 4 ones or 34_{five}:



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* In our decimal numeration system, the base ten is usually not indicated. For example, the number eleven is usually written "11." When various bases are being considered, however, one often does indicate the base ten.

We can display the computation in the same form as for base ten:

$$\begin{aligned} 21_{\text{five}} + 13_{\text{five}} &= (2 \text{ fives} + 1 \text{ one}) + (1 \text{ five} + 3 \text{ ones}) \\ &= (2 \text{ fives} + 1 \text{ five}) + (1 \text{ one} + 3 \text{ ones}) \\ &= 3 \text{ fives} + 4 \text{ ones} \\ &= 34_{\text{five}} \end{aligned}$$

or

$$\begin{array}{r} 21_{\text{five}} = 2 \text{ fives} + 1 \text{ one} \\ \underline{13_{\text{five}} = 1 \text{ five} \ \ + 3 \text{ ones}} \\ 3 \text{ fives} + 4 \text{ ones} = 34_{\text{five}}. \end{array}$$

Remember that we can rearrange addends in a sum because of the commutative and associative properties of addition of whole numbers. Since these are properties of addition and are not dependent in any way on the system of numeration chosen, they may be applied here. These properties, together with the fact that we are still working with a place-value system, permit essentially the same addition algorithm as for the decimal system. Just as in the decimal system, we could use a very abbreviated form:

$$\begin{array}{r} 21_{\text{five}} \\ \underline{13_{\text{five}}} \\ 34_{\text{five}} \end{array}$$

It is very important to keep in mind that the numbers named in the sentence

$$21_{\text{five}} + 13_{\text{five}} = 34_{\text{five}}$$

are not the same as the numbers named in the sentence

$$21_{\text{ten}} + 13_{\text{ten}} = 34_{\text{ten}}.$$

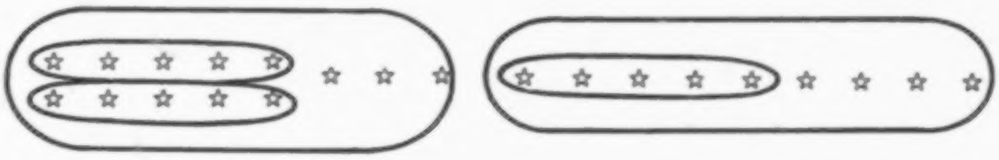
If the base-five sentence is translated into a base-ten sentence, it becomes

$$11_{\text{ten}} + 8_{\text{ten}} = 19_{\text{ten}}.$$

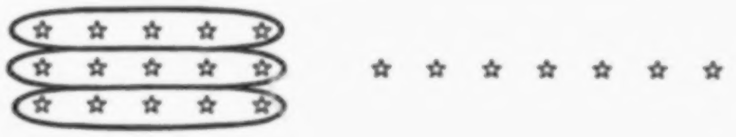
Now let us consider an example requiring regrouping. For the sum

$$23_{\text{five}} + 14_{\text{five}}$$

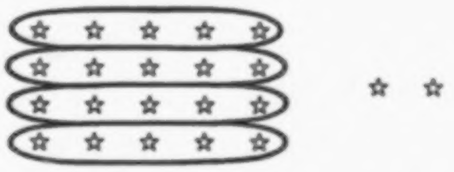
we would think of the union of two sets such as those below.



By grouping the fives and the ones, we obtain this arrangement:



We see that from the set of ones we can form another group of five, which is placed with the other fives:



Since we end up with 4 fives + 2 ones, or 42_{five} objects, we have

$$23_{five} + 14_{five} = 42_{five}.$$

Our computation could assume the following form:

$$\begin{array}{rl} 23_{five} & = 2 \text{ fives} + 3 \text{ ones.} \\ \underline{14_{five}} & \underline{= 1 \text{ five} \ + 4 \text{ ones.}} \\ & = 3 \text{ fives} + (1 \text{ five} + 2 \text{ ones}) \\ & = (3 \text{ fives} + 1 \text{ five}) + 2 \text{ ones} \\ & = 4 \text{ fives} + 2 \text{ ones} \\ & = 42 \end{array}$$

In shorter form we could write:

$$\begin{array}{r} 23_{five} \\ \underline{14_{five}} \\ 42_{five} \end{array}$$

Our thinking would be: "3 ones + 4 ones is 1 five + 2 ones. I'll write down the 2 and remember the 1 five. Then 1 five + 2 fives + 1 five is 4 fives."

Once again, notice that we arrive at this sort of algorithm because of the properties of our number system and the positional nature of the numeration system. No matter what the base is, we are able to use a similar addition algorithm. Of course, we need to use different numeration "facts" in each system. In base five, for example, we need to use such numeration facts as

$$4_{five} + 2_{five} = 11_{five}$$

and

$$3_{five} + 4_{five} = 12_{five}.$$

All the necessary combinations can be entered in a table analogous to that used for base ten.

ADDITION TABLE FOR BASE-FIVE
NUMERATION SYSTEM
(All Entries to Be Interpreted as Base-Five Numerals)

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	10
2	2	3	4	10	11
3	3	4	10	11	12
4	4	10	11	12	13

Let us use this table to compute

$$42_{\text{five}} + 34_{\text{five}}$$

and then we'll check our work by expressing each number in the decimal system.

$$\begin{array}{r} 42_{\text{five}} \\ \underline{34_{\text{five}}} \\ 131_{\text{five}} \end{array}$$

Since

$$42_{\text{five}} = 4 \text{ fives} + 2 \text{ ones} = 22_{\text{ten}}$$

and

$$34_{\text{five}} = 3 \text{ fives} + 4 \text{ ones} = 19_{\text{ten}},$$

our sum should be $22_{\text{ten}} + 19_{\text{ten}}$, or 41_{ten}. Checking, we see that

$$131_{\text{five}} = 1 \text{ twenty-five} + 3 \text{ fives} + 1 \text{ one},$$

which is indeed equal to 41_{ten}.

As a final example, consider the sum

$$203_{\text{five}} + 44_{\text{five}}.$$

$$\begin{array}{r} 203_{\text{five}} \\ \underline{44_{\text{five}}} \\ 302_{\text{five}} \end{array}$$

Check:

$$\begin{aligned} 203_{\text{five}} &= 2 \text{ twenty-fives} + 0 \text{ fives} + 3 \text{ ones} = 53_{\text{ten}}. \\ 44_{\text{five}} &= 4 \text{ fives} + 4 \text{ ones} = 24_{\text{ten}}. \\ 302_{\text{five}} &= 3 \text{ twenty-fives} + 0 \text{ fives} + 2 \text{ ones} = 77_{\text{ten}}, \end{aligned}$$

which is clearly $53_{\text{ten}} + 24_{\text{ten}}$.

Exercise Set 4

1. Use the table above to help you compute the following. Check by converting to the decimal system.

a. $32_{\text{five}} + 43_{\text{five}}$

b. $232_{\text{five}} + 14_{\text{five}}$

c. $132_{\text{five}} + 204_{\text{five}}$

d. $1{,}304_{\text{five}} + 243_{\text{five}}$

2. Construct an addition table for a base-eight numeration system and use it in computing the following sums. Again check by converting to the decimal system.

a. $42_{\text{eight}} + 15_{\text{eight}}$

b. $53_{\text{eight}} + 17_{\text{eight}}$

c. $63_{\text{eight}} + 120_{\text{eight}}$

d. $234_{\text{eight}} + 355_{\text{eight}}$

SUMMARY

Children can use various algorithms for computing sums and computing differences. An algorithm is a step-by-step procedure for renaming a number. As such, it is possible to learn an algorithm by merely memorizing certain facts and certain rules. However, children can have a much more valuable learning experience if they understand an algorithm—that is, if they can apply basic mathematical principles to justify each step in the algorithm.

$a \times (b+c)$

8

$(a \times b) + (a \times c)$

MULTIPLICATION ALGORITHMS AND THE DISTRIBUTIVE PROPERTY

1. **What is the distributive property?**
2. **How do arrays help explain the distributive property?**
3. **What are partial products?**
4. **How do we justify the traditional multiplication algorithm?**

The new mathematics programs strive to emphasize the *ideas* of elementary mathematics. Among these ideas are the whole numbers, the decimal numeration system, basic operations on the whole numbers, and properties of these operations.

Confronted by this emphasis on ideas, teachers may well ask, "What is the place of computation in the new elementary mathematics? Isn't the *how* of arithmetic also important?"

The answer to these questions is that, even in this age of electronic computers, computation is still important in elementary mathematics instruction. The new programs do not overlook the how of arithmetic. However, the techniques, methods, and rules of computation do not stand by themselves; they are by-products of the ideas of mathematics.

Therefore, the new programs strive to make computation processes more meaningful and less mechanical by developing algorithms out of the principles of arithmetic. This chapter will exemplify such a development in the case of multiplication.

Multiplication of whole numbers has four important properties, summarized below:

The associative property of multiplication—

Whenever a, b, and c are whole numbers,

$$(a \times b) \times c = a \times (b \times c).$$

The commutative property of multiplication—

Whenever a and b are whole numbers,

$$a \times b = b \times a.$$

The multiplication property of 1—

Whenever b is a whole number,

$$b \times 1 = b \qquad \text{and} \qquad 1 \times b = b.$$

The multiplication property of 0—

Whenever b is a whole number,

$$b \times 0 = 0 \qquad \text{and} \qquad 0 \times b = 0.$$

These properties are not sufficient to develop an efficient algorithm for computing products of whole numbers. Also needed are the properties of addition of whole numbers which are summarized below:

The associative property of addition—

Whenever a, b, and c are whole numbers,

$$a + (b + c) = (a + b) + c.$$

The commutative property of addition—

Whenever a and b are whole numbers,

$$a + b = b + a.$$

The addition property of 0—

Whenever b is a whole number,

$$b + 0 = b \qquad \text{and} \qquad 0 + b = b.$$

But most important of all to the development of a multiplication algorithm is a property that involves both multiplication and addition. This property, the *distributive* property, is the subject of the early part of this chapter. The rest of the chapter shows how the distributive property, along with the properties of multiplication and addition and the structure of our numeration system, leads to an efficient algorithm for computing products.

THE DISTRIBUTIVE PROPERTY

The following problem is within the scope of a child in about the third grade:

> Five boys sold boxes of Christmas cards. Each boy sold 3 boxes on Friday and 6 boxes on Saturday. How many boxes did the boys sell altogether?

Clearly, there are two approaches to the solution of this problem:

(1) Each boy sold a total of 9 boxes, because $3 + 6 = 9$. Since there were 5 boys and $5 \times 9 = 45$, there were 45 boxes sold altogether.

(2) Fifteen boxes were sold on Friday, since 5 × 3 = 15; and 30 boxes were sold on Saturday, because 5 × 6 = 30. Since 15 + 30 = 45, there were 45 boxes sold altogether.

The computation in these approaches can be summarized as follows:

Approach (1): $5 \times (3 + 6) = 45$.

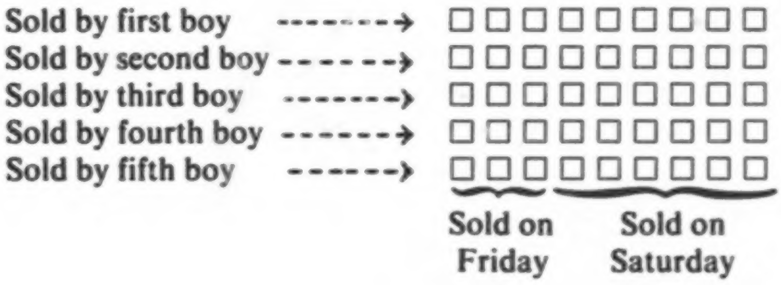
Approach (2): $(5 \times 3) + (5 \times 6) = 45$.

If we wish to express the fact that both approaches produce the same result, we may write:

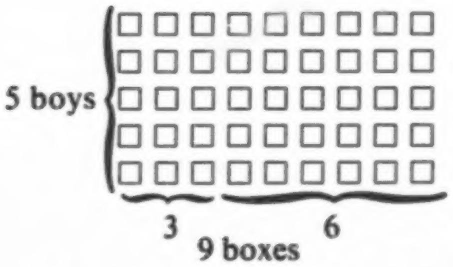
$$5 \times (3 + 6) = (5 \times 3) + (5 \times 6).$$

Without computing the answer, 45, we could have predicted that both approaches would lead to the same result. To show this, let each box of Christmas cards sold by the boys be represented by a square, □.

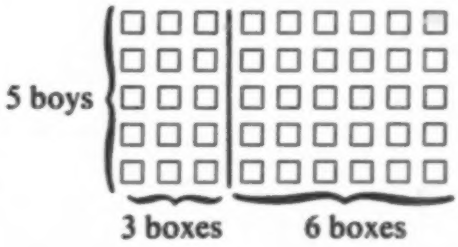
Then all the boxes sold could be represented by an array of squares:


Sold by first boy
Sold by second boy
Sold by third boy
Sold by fourth boy
Sold by fifth boy
Sold on Friday
Sold on Saturday

We may then view this array in two ways:


5 boys
3
6
9 boxes

Viewed as a whole, the array contains 5 × 9, or 5 × (3 + 6), boxes.

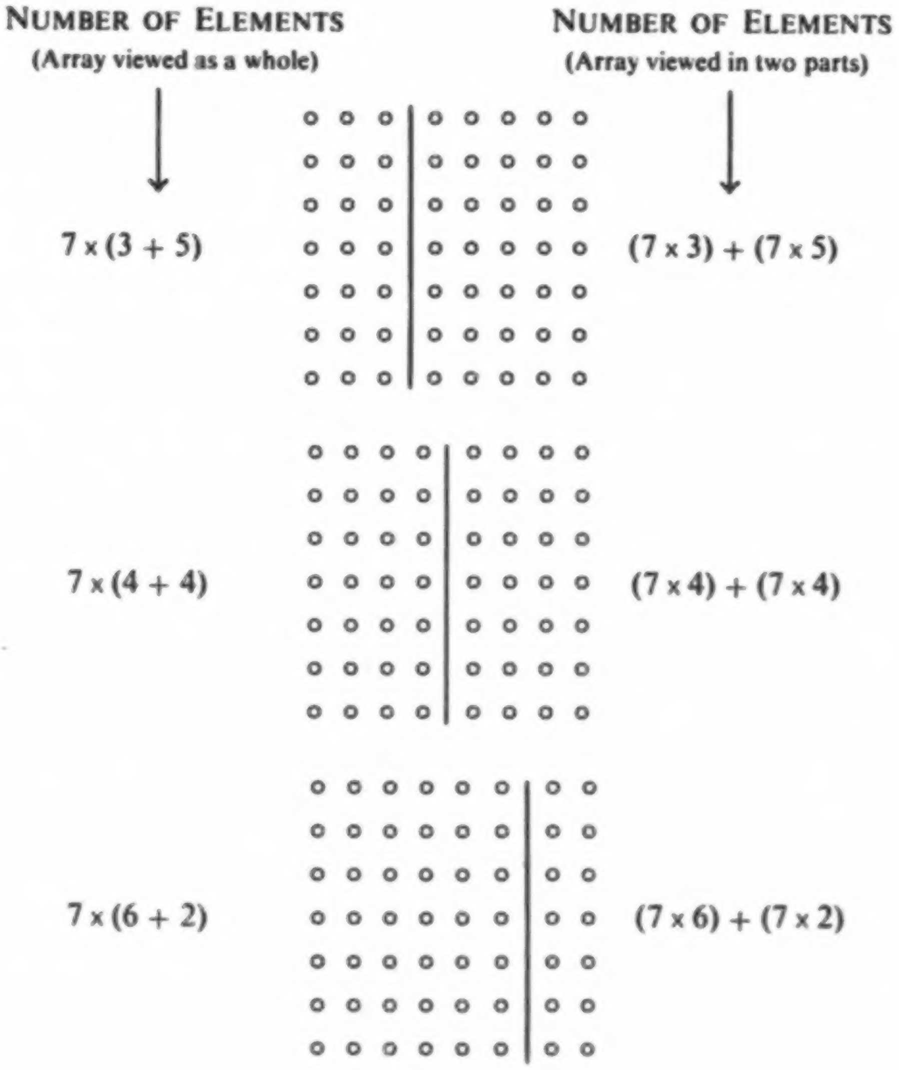

5 boys
3 boxes
6 boxes

Viewed as split into two parts, the array contains (5 × 3) + (5 × 6) boxes.

Since we are merely viewing the *same* array both times, the same number of boxes is represented in each case. That is,

$$5 \times (3 + 6) = (5 \times 3) + (5 \times 6).$$

Any array can be viewed in two ways in the same manner. Consider a 7-by-8 array, which, because of the definition of multiplication, contains 7 × 8 elements. Let us split this array at various places, then express the number of elements in the array accordingly.


NUMBER OF ELEMENTS
(Array viewed as a whole)
NUMBER OF ELEMENTS
(Array viewed in two parts)
7 x (3 + 5)
(7 x 3) + (7 x 5)
7 x (4 + 4)
(7 x 4) + (7 x 4)
7 x (6 + 2)
(7 x 6) + (7 x 2)

Thus, we see that the following statements are true:

$$7 \times (3 + 5) = (7 \times 3) + (7 \times 5).$$
$$7 \times (4 + 4) = (7 \times 4) + (7 \times 4).$$
$$7 \times (6 + 2) = (7 \times 6) + (7 \times 2).$$

These statements and the one shown previously,

$$5 \times (3 + 6) = (5 \times 3) + (5 \times 6),$$

exhibit a pattern. What is this pattern?

$$5 \times (3 + 6) = (5 \times 3) + (5 \times 6).$$

On the left of the equal sign is a product. One of the factors is expressed as a sum.

On the right of the equal sign is a sum. Both of the addends are expressed as products.

$$5 \times (3 + 6) = (5 \times 3) + (5 \times 6).$$

The addends of this sum appear again here.

The general pattern becomes explicit if we let a represent 5, b represent 3, and c represent 6. Then the sentence we have been examining becomes

$$a \times (b + c) = (a \times b) + (a \times c).$$

It is a fact of arithmetic that this pattern holds for all whole numbers a, b, and c. In other words, it is a general property of multiplication and addition that

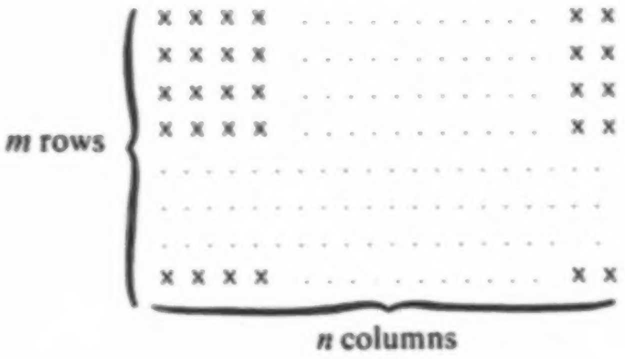
Whenever a, b, and c are whole numbers,

$$a \times (b + c) = (a \times b) + (a \times c).$$

This property is called *the distributive property of multiplication over addition.*

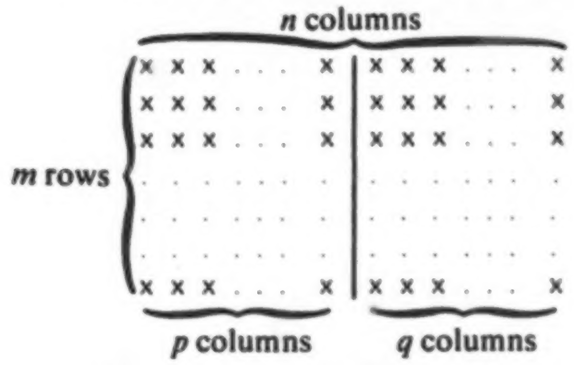
So far we have only looked at examples of this property. Let us now consider a way to show that the property holds for all whole numbers. We merely resort to the same array argument that we used in the example dealing with boys and boxes. However, this time the numbers are arbitrary.

Consider an array with m rows and n columns (m and n can be any whole numbers).



Let us split this array vertically into two parts by thinking of n as a sum,

$p + q$, and then splitting the array into p columns and q columns.



If we look at the array as a whole, the number of elements is

$$m \times (p + q).$$

Multiplying m and $(p + q)$ is the same as multiplying m and n.

If we see the array as split into two parts, the number of elements is

$$(m \times p) + (m \times q).$$

Therefore, for all whole numbers m, p, and q,

$$m \times (p + q) = (m \times p) + (m \times q).$$

This is a formal statement of the distributive property of multiplication over addition.

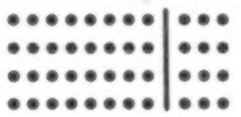
Children may be introduced to the distributive property by trying various numbers for the frames in the sentence

$$\triangle \times (\square + \bigcirc) = (\triangle \times \square) + (\triangle \times \bigcirc).$$

They will find that any numbers for $\triangle$, $\square$, and $\bigcirc$ will make a true statement.

Exercise Set 1

1. Complete these sentences, using the pattern of the distributive property.
   - a. $8 \times (6 + \_\_) = (8 \times 6) + (8 \times 3)$.
   - b. $\_\_ \times (4 + 7) = (6 \times 4) + (\_\_ \times 7)$.
   - c. $3 \times (\_\_ + 5) = (3 \times 6) + (3 \times \_\_)$.
   - d. $6 \times (7 + 8) = (6 \times \_\_) + (\_\_ \times 8)$.
2. Write a sentence of the form $m \times (p + q) = (m \times p) + (m \times q)$ suggested by each of these arrays:
   - a.



$4 \times (8 + 3) = (4 \times 8) + (4 \times 3)$

b. x x x | x x
x x x | x x
x x x | x x ______________________
x x x | x x
x x x | x x

c. ☆ ☆ | ☆
☆ ☆ | ☆
☆ ☆ | ☆
☆ ☆ | ☆ ______________________
☆ ☆ | ☆
☆ ☆ | ☆
☆ ☆ | ☆

d. y y y y y y y y y y | y y y
y y y y y y y y y y | y y y
y y y y y y y y y y | y y y ______________________
y y y y y y y y y y | y y y
y y y y y y y y y y | y y y

3. Draw an array illustrating this example of the distributive property:

$$3 \times (5 + 4) = (3 \times 5) + (3 \times 4).$$

4. Complete these sentences with standard numerals:

a. $5 + 18 =$ ____.

c. $(5 + 2) \times (5 + 9) =$ ____ x ____ = ____.

b. $5 + (2 \times 9) =$ ____.

Is addition distributive over multiplication?

USING THE DISTRIBUTIVE PROPERTY

The distributive property may be used as early as the second grade. Suppose that the pupils are learning multiplication combinations involving 5 as a factor. Suppose also that they already know the multiplication combinations through 4 x 4. How can they learn, for example, that 4 x 5 = 20? They can memorize this combination in rote fashion; they can draw a 4-by-5 array and count it; or they can use the distributive property in the following way:

Since $5 = 2 + 3$,

$$4 \times 5 = 4 \times (2 + 3).$$

Because of the distributive property,

$$4 \times (2 + 3) = (4 \times 2) + (4 \times 3).$$

But

$$(4 \times 2) + (4 \times 3) = 8 + 12, \text{ or } 20.$$

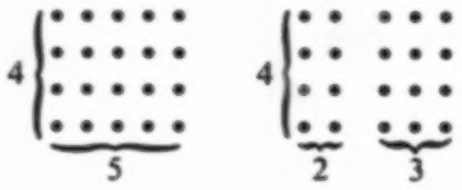
Therefore

$$4 \times 5 = 20.$$

The reasoning used above may be displayed as follows:

$$\begin{aligned} 4 \times 5 &= 4 \times (2 + 3) \\ &= (4 \times 2) + (4 \times 3) \\ &= 8 + 12 \\ &= 20. \end{aligned}$$

This procedure can be further illustrated by utilizing arrays:



$$4 \times 5 = (4 \times 2) + (4 \times 3).$$

In this way, more difficult multiplication facts are made to depend upon easier multiplication facts. If a child temporarily forgets that $4 \times 5 = 20$, he may be able to refresh his memory by using the distributive property.

In the example above the distributive property was used to "obtain," as we say, "$(4 \times 2) + (4 \times 3)$" from "$4 \times (2 + 3)$." We may think of *distributing* "4" *over* "2 + 3":

$$4 \times (2 + 3) = (4 \times 2) + (4 \times 3).$$

This way of thinking of the distributive property, in which "$a \times (b + c)$" is given and "$(a \times b) + (a \times c)$" is obtained, is important in computation. It asserts that these two expressions always name the same number. From the distributive property it follows, for example, that

$$6 \times (4 + 9) = (6 \times 4) + (6 \times 9),$$

so that either expression, "$6 \times (4 + 9)$" or "$(6 \times 4) + (6 \times 9)$," may be used in place of the other.

A "reverse" use of the distributive property to facilitate computation is the following:

What is the sum of 5×97 and 5×3?

We wish to compute $(5 \times 97) + (5 \times 3)$. The straightforward way is to compute $5 \times 97 = 485$ and $5 \times 3 = 15$, then add 485 and 15. An easier way is to observe that the expression $(5 \times 97) + (5 \times 3)$ is in the form $(a \times b) + (a \times c)$. Using the distributive property, we obtain

$$(5 \times 97) + (5 \times 3) = 5 \times (97 + 3).$$

But

$$5 \times (97 + 3) = 5 \times 100, \quad \text{or} \quad 500.$$

So we have

$$(5 \times 97) + (5 \times 3) = 500.$$

The above examples have utilized one *form* of the distributive property—namely, the form $a \times (b + c) = (a \times b) + (a \times c)$. However, we should not rely exclusively on this form. The distributive property can take another form.

Consider the statement

$$9 \times (10 + 7) = (9 \times 10) + (9 \times 7).$$

Let us replace "$9 \times (10 + 7)$" by "$(10 + 7) \times 9$"; both expressions name the same number because multiplication is commutative. Also, let us replace "9×10" by "10×9" and "9×7" by "7×9." Then the statement

$$(10 + 7) \times 9 = (10 \times 9) + (7 \times 9)$$

is true. The form of this statement is

$$(b + c) \times a = (b \times a) + (c \times a).$$

We should also learn this form of the distributive property. In fact, this form will arise in our later work with multiplication computation.

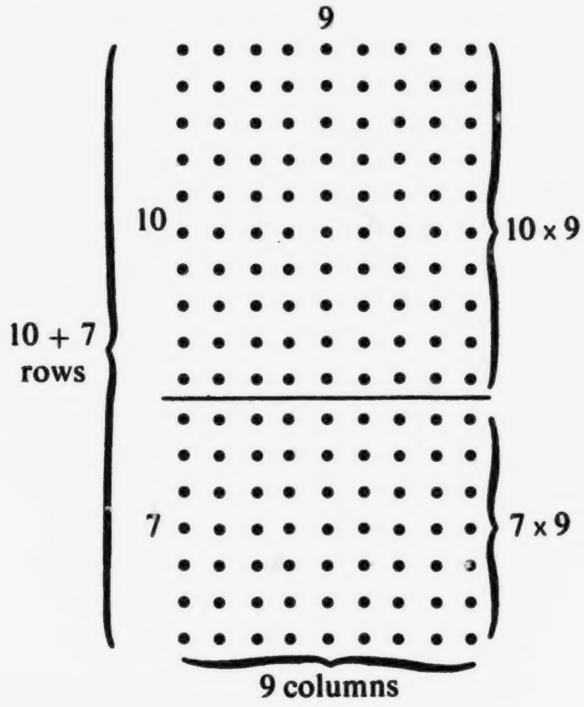
An array to illustrate the form

$$(b + c) \times a = (b \times a) + (c \times a)$$

for the example

$$(10 + 7) \times 9 = (10 \times 9) + (7 \times 9)$$

is the following:



Memorization of multiplication combinations usually ceases at $9 \times 9 = 81$. How do we compute products when one of the factors exceeds 9? We can use the distributive property to good advantage.

EXAMPLE: Compute the product of 16 and 7.

Solution 1: We may rename the larger factor, 16, as a sum of numbers less than 10—let us say $9 + 7$.

$$16 \times 7 = (9 + 7) \times 7.$$

We then use the distributive property,

$$(9 + 7) \times 7 = (9 \times 7) + (7 \times 7);$$

and, if we know the combinations through $9 \times 9 = 81$, we find that

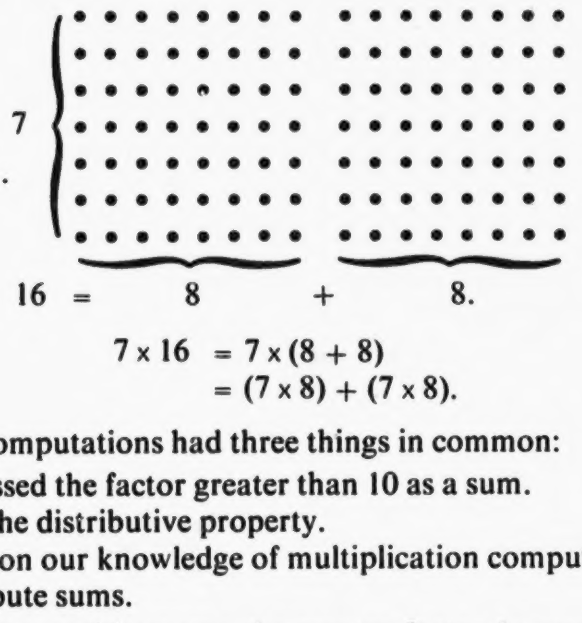
$$(9 \times 7) + (7 \times 7) = 63 + 49, \quad \text{or} \quad 112.$$

Therefore,

$$16 \times 7 = 112.$$

Solution 2: We may express the product 16×7 as 7×16. We can rename 16 in various ways as a sum of numbers less than 10. For example, we can say $8 + 8$.

$$\begin{aligned} 7 \times 16 &= 7\,(8 + 8) && \text{(Renaming 16 as “8 + 8”)}\\ &= (7 \times 8) + (7 \times 8) && \text{(Using the distributive property)}\\ &= 56 + 56 && \text{(Renaming } 7 \times 8 \text{ as “56”)}\\ &= 112. && \text{(Addition computation)} \end{aligned}$$



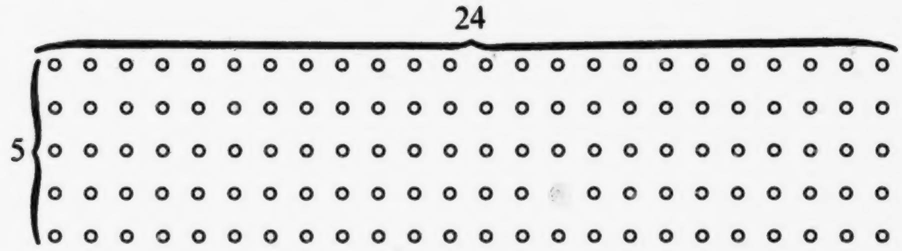
$$\begin{aligned} 7 \times 16 &= 7 \times (8 + 8)\\ &= (7 \times 8) + (7 \times 8). \end{aligned}$$

These two computations had three things in common:

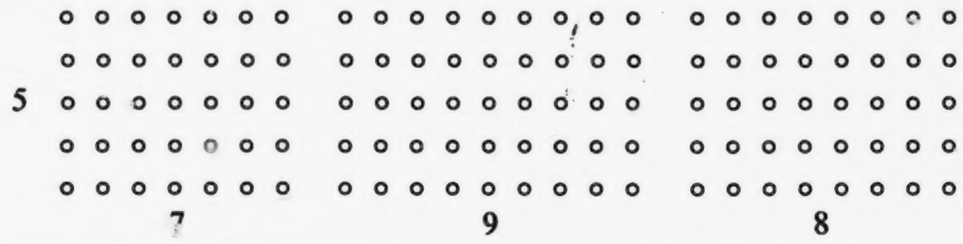
1. We expressed the factor greater than 10 as a sum.
2. We used the distributive property.
3. We relied on our knowledge of multiplication computation and our ability to compute sums.

Notice that we could rename the greater factor in several ways and we could use the distributive property in either of two forms.

A further extension of the distributive property will allow us to compute products such as 5×24, relying only on combinations through $9 \times 9 = 81$. Let us look at a 5-by-24 array:



If we split the array vertically into two parts, the number of columns in at least one part will still exceed 10. So let us split the array into three parts:



This arrangement is equivalent to renaming 24 as "7 + 9 + 8." Splitting the array shows that

$$5 \times (7 + 9 + 8) = (5 \times 7) + (5 \times 9) + (5 \times 8).$$

We can then proceed, as before, to compute the sum 35 + 45 + 40, or 120, as the product 5×24.

The technique used in this section for computing products will be extended and refined in the succeeding sections.

Exercise Set 2

1. Which property assures that the following statements are true? Write D for the distributive property, A for the associative property of multiplication, and C for the commutative property of multiplication.

a. $3 \times (4 + 5) = (3 \times 4) + (3 \times 5)$.___

b. $3 \times (4 \times 5) = (4 \times 5) \times 3$.___

c. $(4 + 5) \times 3 = (4 \times 3) + (5 \times 3)$.___

d. $(4 \times 3) + (5 \times 3) = (5 \times 3) + (4 \times 3)$.___

e. $3 \times (4 \times 5) = (3 \times 4) \times 5$.___

2. Use the distributive property to compute the following products. (Assume that the multiplication facts through $9 \times 9 = 81$ are the only ones known.)

a. $6 \times 14 = 6 \times (9 + 5) = (6 \times 9) + (6 \times 5) = 54 + 30 = 84.$

b. $7 \times 15 =$

c. $13 \times 8 =$

d. $26 \times 6 =$

3. Use the distributive property to find a standard numeral for each of these expressions:

a. $(18 \times 3) + (2 \times 3) = (18 + 2) \times 3 = 20 \times 3 = 60.$

b. $(98 \times 2) + (2 \times 2) =$

c. $(6 \times 189) + (6 \times 11) =$

AN ALGORITHM FOR MULTIPLICATION

To compute the product of two whole numbers means to find the standard numeral for the product. For example, to compute the product of 35 and 6 means to find the numeral "210" to name 35×6.

Why should we be interested in a systematic procedure, or algorithm, for computing products? Because we want to be able to compute the product of *any* two whole numbers, and we want to perform the computation with reasonable speed and accuracy.

In the previous section we showed how the distributive property can be used to compute products such as 7×14 (that is, products in which one factor is less than 10). If the computation of 7×14 is performed by three of your pupils, they may all do it differently. For example:

Susan

$$\begin{aligned} 7 \times 14 &= 7 \times (5 + 9) \\ &= (7 \times 5) + (7 \times 9) \\ &= 35 + 63 \\ &= 98. \end{aligned}$$

John

$$\begin{aligned} 7 \times 14 &= 7 \times (8 + 6) \\ &= (7 \times 8) + (7 \times 6) \\ &= 56 + 42 \\ &= 98. \end{aligned}$$

Mary

$$\begin{aligned} 7 \times 14 &= 7 \times (7 + 7) \\ &= (7 \times 7) + (7 \times 7) \\ &= 49 + 49 \\ &= 98. \end{aligned}$$

These three computations differ merely in the way in which the greater factor is renamed. Among the various possible ways to rename 14, is there a particularly convenient way? The easiest renaming of 14 makes use of the meaning of the numeral "14," namely, that "14" means 1 ten and 4 ones.

$$14 = 10 + 4.$$

So, to compute 7 x 14, we write

$$7 \times 14 = 7 \times (10 + 4)$$

and, using the distributive property,

$$7 \times (10 + 4) = (7 \times 10) + (7 \times 4).$$

Now, even if combinations through 10's are not memorized, we know that 7 x 10 is 7 tens, which is 70. So

$$(7 \times 10) + (7 \times 4) = 70 + 28 = 98.$$

Let us try this same technique to compute 8 x 56.

$$8 \times 56 = 8 \times (50 + 6).$$

We rename 56 as "50 + 6," because 50 + 6 is an "expanded form" of 56. Then we use the distributive property:

$$8 \times (50 + 6) = (8 \times 50) + (8 \times 6).$$

Now we need to compute 8 x 50. We may use the associative property of multiplication:

$$\begin{aligned} 8 \times 50 &= 8 \times (5 \times 10) \\ &= (8 \times 5) \times 10 \\ &= 40 \times 10 \\ &= 400. \end{aligned}$$

But children will soon learn that to compute 8 x 50 they need only compute 8 x 5 = 40 and attach the digit "0" to the numeral "40," thus obtaining "400."

So the complete computation of 8 x 56 would appear as follows:

$$\begin{aligned} 8 \times 56 &= 8 \times (50 + 6) \\ &= (8 \times 50) + (8 \times 6) \\ &= 400 + 48 \\ &= 448. \end{aligned}$$

As long as we restrict one factor to the numbers less than 10, we can use this technique for any multiplication computation, because any whole number can be named in "expanded form." For example:

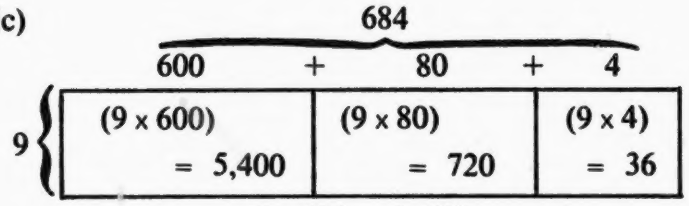
To compute 8 x 72, we rename 72 as "70 + 2."
To compute 6 x 146, we rename 146 as "100 + 40 + 6."
To compute 9 x 684, we rename 684 as "600 + 80 + 4."

Then we proceed to apply the distributive property. Clearly, our place-value numeration system plays a vital role in this method of computation, just as it did in addition and subtraction computations.

When the greater factor has 3 digits, as in the product 9 x 684, we extend the distributive property as we did on page 119 of this chapter.

$$\begin{aligned} 9 \times 684 &= 9 \times (600 + 80 + 4) \\ &= (9 \times 600) + (9 \times 80) + (9 \times 4) \\ &= 5{,}400 + 720 + 36 \\ &= 6{,}156. \end{aligned}$$

(Schematic)



9 x 684 = 6,156.

Computations of this type can be arranged in a "vertical form." Consider 9 x 684 again. We may write:

Method A

$$\begin{array}{r} 600 + 80 + 4 \\ \times \quad 9 \\ \hline 5{,}400 + 720 + 36 = 6{,}156 \end{array}$$

or

Method B

$$\begin{array}{r} 4 + 80 + 600 \\ \times \quad 9 \\ \hline 36 + 720 + 5{,}400 = 6{,}156. \end{array}$$

Since an addition computation is required after using the distributive property, it is convenient to list 5,400 and 720 and 36 vertically:

Method A

$$\begin{array}{r} 684 \\ \times\ 9 \\ \hline 5{,}400 \\ 720 \\ 36 \\ \hline 6{,}156 \end{array}$$

or

Method B

$$\begin{array}{r} 684 \\ \times\ 9 \\ \hline 36 \\ 720 \\ 5{,}400 \\ \hline 6{,}156 \end{array}$$

When using this vertical arrangement, we mentally rename 684 as 600 + 80 + 4, then use the distributive property. In Method B, 9 x 4 = 36 is computed first, then 9 x 80 = 720, then 9 x 600 = 5,400. We may now pass to a shortcut for the whole process. The shortcut may be described as follows:

STEP 1.—Write "684," and "9" vertically below it. Mentally rename 684 as "600 + 80 + 4." Compute 9 x 4 = 36, but write only the "6" of the "36" in the ones column. *Remember* the 3 tens of 36.

$$\begin{array}{r} 684 \\ \times\ 9 \\ \hline 6 \end{array}$$

STEP 2.—Compute 9 x 80 = 720. Add the 3 tens, or 30, to 720 to obtain 750. Write "5" in the tens column and *remember* 7 hundreds.

$$\begin{array}{r} 684 \\ \times\ 9 \\ \hline 56 \end{array}$$

STEP 3.—Compute 9 x 600 = 5,400. Add the 7 hundreds to 5,400 to obtain 6,100. Write "1" in the hundreds column and "6" in the thousands column.

$$\begin{array}{r} 684 \\ \times\ 9 \\ \hline 6{,}156 \end{array}$$

This final shortcut, if presented as the only way to compute products, could easily obscure the use of the expanded form of the distributive property. Therefore, it should be presented as the *last* in a sequence of forms for recording the computation of products.

In the next section we shall remove the restriction that one factor be less than 10 and thus develop an algorithm for computing the product of any two whole numbers.

Exercise Set 3

1. Express each of the following numbers in an expanded form:

a. 157 = 100 + 50 + 7.

b. 259 =

c. 35 =

d. 4,560 =

e. 408 =

2. Complete these sentences by renaming the larger factor in an expanded form and by using the distributive property.

a. 7 x 16 = 7 x (10 + __)
= (7 x __) + (7 x __)
= 70 + __
= ___.

b. 5 x 97 = 5 x (__ + __)
= (5 x __) + (5 x __)
= __ + 35
= ___ .

c. 3 × 655 = ____ × (600 + 50 + 5)

= (____ × 600) + (____ × 50) + (____ × 5)

= ____ + ____ + ____

= ____ .

3. Identify parts of these computations by filling in the blanks.

a.

$$\begin{array}{r l} 147 & \\ \underline{\times\ 4} & \\ 28 & \leftarrow 4 \times 7 \\ 160 & \leftarrow 4 \times \underline{\quad} \\ \underline{400} & \leftarrow \underline{\quad} \times \underline{\quad} \\ 588 & \leftarrow 4 \times \underline{\quad} \end{array}$$

b.

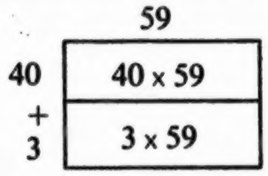
$$\begin{array}{r l} 1{,}508 & \\ \underline{\times\ \ 3} & \\ 24 & \leftarrow 3 \times \underline{\quad} \\ 1{,}500 & \leftarrow \underline{\quad} \times \underline{\quad} \\ \underline{3{,}000} & \leftarrow \underline{\quad} \times \underline{\quad} \\ 4{,}524 & \leftarrow \underline{\quad} \times \underline{\quad} \end{array}$$

EXTENDING THE MULTIPLICATION ALGORITHM

We now extend the multiplication algorithm to computation of products such as 43 × 59, 43 × 257, 125 × 356, etc. We do not restrict the size of either factor. We shall again depend upon the place-value property of our numerals and the distributive property. Moreover, we shall assume the results of the previous section—namely, an algorithm for computing a product when one of two factors is less than 10.

Let us proceed step by step to compute the product 43 × 59.

We rename the first factor, 43, in expanded form:



$$43 \times 59 = (40 + 3) \times 59.$$

Then we distribute the other factor, 59, over the sum 40 + 3. We shall use the distributive property in the form

$$(b + c) \times a = (b \times a) + (c \times a),$$

obtaining

$$(40 + 3) \times 59 = (40 \times 59) + (3 \times 59).$$

Now we rename 40 as 10 × 4 to obtain

$$(40 \times 59) + (3 \times 59) = (10 \times 4 \times 59) + (3 \times 59).$$

Notice that we now need to compute 4 × 59 and 3 × 59. These computations are the kind explained in the previous section, so we assume they can be performed.

$$\begin{array}{r} 59 \\ \underline{\times\ 4} \\ 236 \end{array} \qquad \qquad \begin{array}{r} 59 \\ \underline{\times\ 3} \\ 177 \end{array}$$

We then arrive at this statement:

$$(10 \times 4 \times 59) + (3 \times 59) = (10 \times 236) + (177).$$

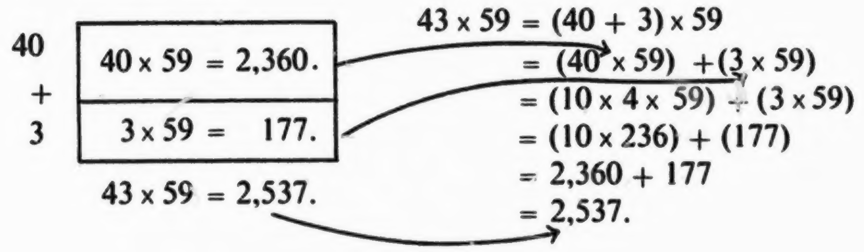
We also assume that a method for multiplying by 10 is known.

$$\begin{aligned}(10 \times 236) + 177 &= 2{,}360 + 177 \\ &= 2{,}537.\end{aligned}$$

Therefore,

$$43 \times 59 = 2{,}537.$$

The entire computation of 43 x 59 may be displayed as follows:



$$\begin{aligned}43 \times 59 &= (40 + 3) \times 59 \\ &= (40 \times 59) + (3 \times 59) \\ &= (10 \times 4 \times 59) + (3 \times 59) \\ &= (10 \times 236) + (177) \\ &= 2{,}360 + 177 \\ &= 2{,}537.\end{aligned}$$

Notice that the two multiplication computations that had to be done along the way were 4 x 59 = 236 and 3 x 59 = 177. If we write 43 and 59 in vertical form, we can easily perform the two intermediate computations. We first compute 3 x 59 and write 177.

Then we compute 4 x 59, but we write 2,360 (since 4 x 59 is also multiplied by 10).

We then add 177 and 2,360.

$$\boxed{\begin{array}{rr} 59 & 59 \\ \underline{\times\ 43} & \underline{\times\ 43} \\ 177 & 177 \\ & \underline{2{,}360} \\ & 2{,}537 \end{array}}$$

The extension of this algorithm to factors with three or more digits does not involve any new ideas, but the computation gets longer and more complicated.

Computation of products such as 43 x 59 may be performed by a longer method than the one we have so far explored. We begin as before:

$$\begin{aligned}43 \times 59 &= (40 + 3) \times 59 \\ &= (40 \times 59) + (3 \times 59).\end{aligned}$$

But now 59 is also renamed in expanded form,

$$43 \times 59 = [40 \times (50 + 9)] + [3 \times (50 + 9)],$$

and the 40 and 3 are distributed over the 50 + 9:

$$43 \times 59 = (40 \times 50) + (40 \times 9) + (3 \times 50) + (3 \times 9).$$

At this stage, pupils will need to know how to compute these products:

40 x 50 = 2,000,
40 x 9 = 360,
3 x 50 = 150,
and
3 x 9 = 27.

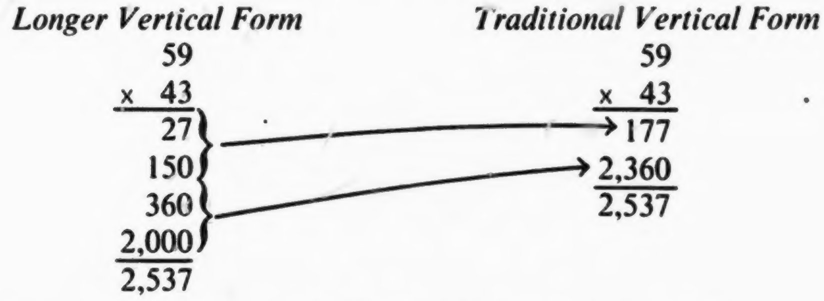
	50	+ 9
40 +	40 x 50 = 2,000.	40 x 9 = 360.
3	3 x 50 = 150.	3 x 9 = 27.

The computation of 40 x 50 = 2,000, for example, can be carried out by rearranging factors:

$$\begin{aligned} 40 \times 50 &= (40 \times 10) \times (5 \times 10) \\ &= (4 \times 5) \times (10 \times 10) \\ &= 20 \times 100 \\ &= 2{,}000. \end{aligned}$$

$$\begin{aligned} 43 \times 59 &= 2{,}000 + 360 + 150 + 27 \\ &= 2{,}537. \end{aligned}$$

The traditional algorithm combines certain partial products as follows:


Longer Vertical Form

Traditional Vertical Form

The longer vertical form can be valuable in teaching a multiplication algorithm to slower students. It can also be used profitably with all students to review the principles underlying the traditional multiplication algorithm.

Exercise Set 4

1. Complete these sentences:

a. 21 x 32 = (20 + 1) x 32
= (20 x ___) + (1 x ___)
= (10 x ___ x 32) + (1 x ___)
= (10 x ___) + ___
= ___ + ___
= ___.

b. 34 x 156 = (30 + ___) x ___

= (30 x ___) + (___ x 156)

= (___ x 3 x 156) + (___ x 156)

= (___ x 468) + ___

= ___ + ___

= _____.

2. Draw an array to depict

$$16 \times 12 = (10 \times 10) + (10 \times 2) + (6 \times 10) + (6 \times 2).$$

SUMMARY

We have used the properties of multiplication and addition and the structure of our numeration system to develop methods of computing products. The property we leaned upon most heavily is *the distributive property of multiplication over addition*:

For all whole
numbers, a, b, and c,
$$a \times (b + c) = (a \times b) + (a \times c).$$

We used the structure of our numeration system by renaming factors in expanded form.

The properties of multiplication and addition, commutativity and associativity, were used whenever we rearranged factors or addends.

To show how many of these properties are used in a single computation, we shall compute the product 21 x 308 in detail and explain the justification for each step:

21 x 308 = (20 + 1) x 308	Naming 21 in expanded form makes use of the place-value idea of our numeration system.
= (20 x 308) + (1 x 308)	This step is an application of the distributive property, in a form arrived at through the commutative property of multiplication.
= (10 x 2 x 308) + (1 x 308)	Naming 20 as 10 x 2 uses the numeration system. Writing 3 factors (10, 2, and 308) without parentheses uses the associative property of multiplication.

= (10 x 616) + 308	Computing 1 x 308 = 308 uses the multiplication property of 1. Computing 2 x 308 uses several properties.
= 6,160 + 308	Computing 10 x 616 uses the structure of the numeration system.
= 6,468.	The addition algorithm uses several properties of addition.

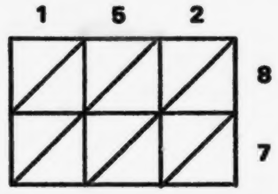
All of this reasoning occurs "behind the scenes" when we write the short form of the multiplication algorithm.

$$\begin{array}{r} 308 \\ \underline{\times\ 21} \\ 308 \\ \underline{6{,}160} \\ 6{,}468 \end{array}$$

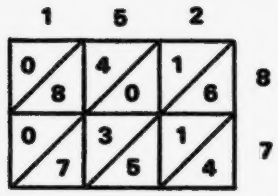
This shortcut is brief, efficient, convenient to write and print. It has been in use since the fifteenth century. However, the justification we have given for the method, in terms of properties of multiplication, addition, and numeration, was not provided until the nineteenth century.

The algorithm developed in this lesson is certainly not the only algorithm for multiplication of whole numbers. It is popular today because of its brevity and convenience. Other algorithms have been used, but have become unpopular for various reasons. One algorithm, often called the net, or grating method, and also dating to the fifteenth century, is described below. According to D. E. Smith, "the method is very old and might have remained the popular one if it had not been difficult to print or even to write the net."

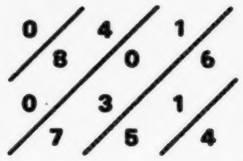
Let us see how to compute 87 x 152 by the grating method. We begin by drawing the grating and writing the factors as follows:



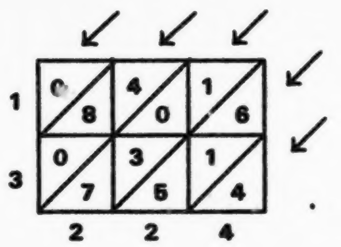
Each digit is treated as a single number regardless of its place value. The product of 8 and 2 is recorded in the square where the 2 column and the 8 row intersect; the tens digit of 16 is placed above the diagonal, the ones digit below the diagonal. We continue in this fashion, multiplying 8 by 5, 8 by 1, 7 by 2, 7 by 5, and 7 by 1, always writing the tens digit above the appropriate diagonal and the ones digit below.



Then we shift our attention to the digits in the grating and regard them as being in slanted columns.



We add the numbers named in these slanted columns, beginning at the lower right and "carrying" from column to column.



The product is then named by using the numerals appearing at the left edge and at the bottom of the grating.

87 x 152 = 13,224.

What properties do you think are needed to justify the grating method?

Any algorithm requires practice to be useful, and elementary school pupils will need to spend a large amount of time practicing the multiplication algorithm that we teach today. If pupils learn why the algorithm works and what the steps accomplish, their practice in computing will be less trying.

9

DIVISION ALGORITHMS

1. **What interpretation of "quotient" is a basis for the traditional division algorithm?**
2. **What is a partial quotient?**
3. **How does the distributive property help justify the traditional algorithm?**
4. **What important skills are needed in order to use the traditional division algorithm efficiently?**
5. **As compared with the traditional division algorithm, what are the advantages of the algorithm in which the partial quotients are listed vertically?**

ELEMENTARY APPROACHES TO DIVISION

If you gave your pupils the following problem, how would you expect them to solve it?

EXAMPLE 1: A teacher has 12 chocolates that she plans to give to some children. If each child will get 3 pieces, how many children can receive candy?

Your expectations, of course, would depend upon the previous experience of the pupils with this type of problem. Let's begin with the simplest technique a child could use and then develop other approaches to the solution of the problem.

A child's first approach to a situation of this type would probably be to separate the set of 12 chocolates into equivalent subsets,* each subset containing 3 pieces.

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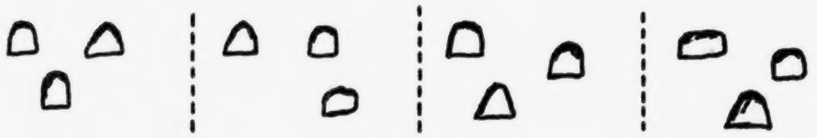
* This terminology need not be used with younger pupils.



Upon checking the results, the child determines that he has 4 groups; thus, the candy would be given to 4 children.

When students reach the stage at which they are ready to develop an algorithm for division, they can resort to their knowledge of subtraction. An algorithm that most closely resembles the use of equivalent subsets is given below:

EXAMPLE 1 (a second approach):

$$\begin{array}{rcl}
12 & = & \text{number of elements in the original group.}\\
\underline{-\ 3} & 1 & \\
9 & & \text{How many threes can we subtract,}\\
\underline{-\ 3} & 1 & \text{starting with 12? Four.}\\
6 & & \\
\underline{-\ 3} & 1 & \\
3 & & \\
\underline{-\ 3} & \underline{1} & \\
0 & 4 &
\end{array}$$

Let us try this method on another problem.

EXAMPLE 2: Jim has 30¢ to spend on apples. If each apple costs 6¢, how many apples can he buy?

Solution:

$$\begin{array}{rcl}
30 & & \\
\underline{-\ 6} & 1 & \text{How many sixes can we}\\
24 & & \text{subtract, starting with}\\
\underline{-\ 6} & 1 & \text{30? Five.}\\
18 & & \\
\underline{-\ 6} & 1 & \\
12 & & \\
\underline{-\ 6} & 1 & \\
6 & & \\
\underline{-\ 6} & \underline{1} & \\
0 & 5 &
\end{array}$$

The method used for solving Examples 1 and 2 is usually called the repeated-subtraction procedure. Why are we able to obtain the answer to a division problem by repeated subtraction? At the physical level, division is related to finding how many equivalent subsets of a given size can be formed from a given set. Therefore, one interpretation of division is analogous, at the number level, to subtracting the divisor repeatedly.

The process of repeated subtraction, adequate as an introductory algorithm, is a relatively crude method. However, as we develop more efficient algorithms, we shall see that they are modifications of the process of repeated subtraction. Even when working with large numbers we are using a refined form of repeated subtraction.

DIVISION RELATED TO MULTIPLICATION

Another approach to developing an efficient division algorithm is based on the relationship between multiplication and division.

Multiplication Situation

EXAMPLE 3: While looking through his stamp book, Jim notices that on a certain page he has 5 rows of stamps with each row containing 6 stamps. How many stamps are on this page?

This problem could be expressed in the following manner:

$$\underset{\substack{\downarrow\\ \text{factor}}}{5} \times \underset{\substack{\downarrow\\ \text{factor}}}{6} = \underset{\substack{\downarrow\\ \text{missing}\\ \text{product}}}{\square}.$$

Division Situation

EXAMPLE 4: Jim bought 30 stamps that are to be placed on a certain page in his stamp book. He wishes to have 5 rows of stamps on this page. How many stamps will he need to place in each row?

If we think of a division situation as one in which we determine the missing factor, this problem can be expressed in the following manner:

$$\underset{\substack{\downarrow\\ \text{factor}}}{5} \times \underset{\substack{\downarrow\\ \text{missing}\\ \text{factor}}}{\square} = \underset{\substack{\downarrow\\ \text{product}}}{30}.$$

The child's knowledge of his multiplication combinations will enable him to determine the missing factor.

$$5 \times \boxed{6} = 30.$$

The mathematical sentence

$$5 \times \square = 30$$

can also be written in the form

$$\square = 30 \div 5.$$

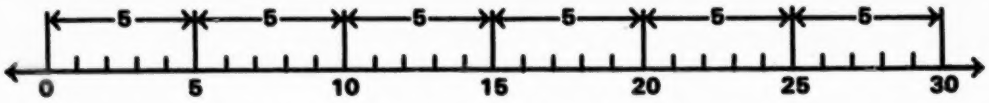
The two mathematical sentences above show an inverse relationship of multiplying by 5 and dividing by 5.

In the solution of Example 4, the missing factor, 6, is the quotient of the pair of numbers 30 and 5. In a division problem, no matter what process you use for determining the quotient of two numbers, you are basically finding the missing factor.

Another way to deal with a division problem is to give it a geometric interpretation. Let us consider the following example:

EXAMPLE 5: The seniors need ribbon badges to identify ushers for their class play. They decide to make each badge 5 inches long. If they buy 30 inches of ribbon, how many badges can they make?

This situation can be represented on a number line as follows:



The diagram shows that exactly six 5-inch badges can fit end to end, thereby showing that the answer to the problem

$$\underset{\text{missing factor}}{\underset{\downarrow}{\square}} \times 5 = 30 \quad \text{is } \boxed{6} \times 5 = 30, \quad \text{or} \quad \boxed{6} = 30 \div 5.$$

DIVISION MAY NOT BE POSSIBLE WITH WHOLE NUMBERS

Thus far we have been working with pairs of whole numbers for which it is possible to determine the missing factor. Now let's examine a situation in which this is not possible. Suppose Example 1 had stated that the teacher had 13 chocolates. Using repeated subtraction, we have

$$\begin{array}{r} 13 \\ -\ 3 \\ \hline 10 \\ -\ 3 \\ \hline 7 \\ -\ 3 \\ \hline 4 \\ -\ 3 \\ \hline 1 \end{array}$$

We have subtracted 3 four times, but we have 1 chocolate left over.

The physical solution and the repeated-subtraction process yield similar results whether we begin with 12 or with 13 chocolates. But when we express the situation in terms of a missing factor, dealing with 12 chocolates is different from dealing with 13 chocolates. For when we start with 12 chocolates, it is possible to obtain a whole number for the missing factor:

$$\underset{\substack{\downarrow \\ \text{missing} \\ \text{factor}}}{\square} \times 3 = 12;$$

that is,

$$\boxed{4} \times 3 = 12.$$

But when we have 13 chocolates, there is no whole number that will satisfy the condition

$$\square \times 3 = 13.$$

In general terms, if a and b are whole numbers, then the mathematical sentence $\square \times a = b$ cannot always be "satisfied" by a whole number. But it is always possible to find a *pair* of whole numbers that will satisfy the following sentence for any given whole numbers a and b:

$$(\square \times a) + \triangle = b.$$

In fact, we can usually find several such pairs. For example, in the case of the 13 chocolates, we could write the following mathematical sentences:

$$(\boxed{4} \times 3) + \overset{1}{\triangle} = 13, \quad \text{or}$$
$$(\boxed{2} \times 3) + \overset{7}{\triangle} = 13.$$

Let's look at another situation of this type.

EXAMPLE 6: Five girls plan to share *equally* a half-pound of cookies that they have bought. When they count the cookies they find that there are 33 cookies. What is the maximum number of cookies each girl can receive?

A mathematical sentence that attempts to express this problem is

$$33 = \square \times 5.$$

But there is no whole number that will satisfy this sentence. However, if we express our problem with the mathematical sentence $33 = (\square \times 5) + \triangle$, the following relationship does hold:

$$33 = (\boxed{6} \times 5) + \overset{3}{\triangle}.$$

Interpreting this mathematical sentence in terms of the problem, we say that each girl could receive 6 cookies and there would be 3 extra cookies.

The following example illustrates one way to develop an understanding of division computation involving a remainder.

EXAMPLE 7: Determine a pair of whole numbers that will make the following sentence true:

$$34 = (\square \times 7) + \triangle.$$

When we examine this sentence, we find that there are several pairs of whole numbers that will satisfy the sentence.

(1) $34 = (\boxed{0} \times 7) + \triangle{34}.$

(2) $34 = (\boxed{1} \times 7) + \triangle{27}.$

(3) $34 = (\boxed{2} \times 7) + \triangle{20}.$

(4) $34 = (\boxed{3} \times 7) + \triangle{13}.$

(5) $34 = (\boxed{4} \times 7) + \triangle{6}.$

The answer in (5) is the one having the smallest whole number that can be represented in the triangle and the largest one in the square. [Note that each number represented in the triangle is greater than 7 except for example (5).]

In special cases, the smallest whole number for the $\triangle$ is 0. For example:

$$35 = (\boxed{5} \times 7) + \triangle{0}.$$

In this case, 5 and 7 are factors of 35. This corresponds to the situation previously treated.

Exercise Set 1

1. For each mathematical sentence given below, determine a pair of whole numbers that will make the sentence a true statement. Try to determine a pair of numbers such that the number represented in the triangle will be the least possible (and the number in the square, therefore, the greatest).

a. $23 = (\square \times 5) + \triangle.$

b. $5 = (\square \times 7) + \triangle.$

c. $6 = (\square \times 6) + \triangle.$

d. $97 = (\square \times 9) + \triangle.$

e. $38 = (\square \times 6) + \triangle.$

f. $31 = (\square \times 4) + \triangle.$

g. $0 = (\square \times 8) + \triangle.$

h. $61 = (\square \times 20) + \triangle.$

NOTE.—The frames are helpful in mathematical sentences of this type, particularly for the younger pupils. Later, another way of writing these sentences will be shown—a way that can be used with older students.

2. After solving problems 1a - 1h, examine the number represented in the triangle. If you are successful in determining the least whole number for the triangle that makes the sentence true, what relationship does this number have to the given factor?

3. For those problems (1a - 1h) in which you were able to determine the least whole number to be represented in the triangle, see how many different numbers you can represent in the square. For example, in the problem 1a you should have found that 3 is the least number to be represented in the triangle.

$$23 = (\boxed{4} \times 5) + \stackrel{3}{\triangle}.$$

Retaining 3 for the triangle,

$$23 = (\square \times 5) + \stackrel{3}{\triangle},$$

is it possible to determine values other than 4 for the square so that we still have a true mathematical sentence?

EXTENDING THE DIVISION IDEA

What do relationships of the type given in Exercise Set 1 have to do with division? In Example 6 (the problem of sharing the 33 cookies), we found that the symbol $33 \div 5$ has no meaning in the set of whole numbers. That is, the following condition cannot be satisfied by a whole number:

$$33 = \square \times 5.$$

But we can form the following true statement:

$$33 = (6 \times 5) + 3.$$

This sentence describes an extension of the ideas of division. Of course, if we had fractional numbers at our disposal, we could define the operation of division in terms of a missing factor. But since we are working with whole numbers only, we must extend our idea of division and express it in the following way:

> *For any pair of whole numbers a and b, where $b \neq 0$, it is always possible to determine a pair of whole numbers q (quotient) and r (remainder) such that*
>
> $$a = (q \times b) + r, \quad \textit{and} \quad r < b.$$
>
> a: dividend $\qquad$ b: divisor
>
> q: quotient $\qquad$ r: remainder

(By stipulating that r must be less than b, we find the greatest q possible.)

Let's see how we might use the above relationship to solve a problem.

EXAMPLE 8: Dick has 61 stamps to be placed on a certain page of his stamp book. He wishes to arrange them in rows so that there will be 8 stamps in each row. How many complete rows of 8 can he have? How many stamps will remain?

Solution: The following sentence describes the relationship in this problem:

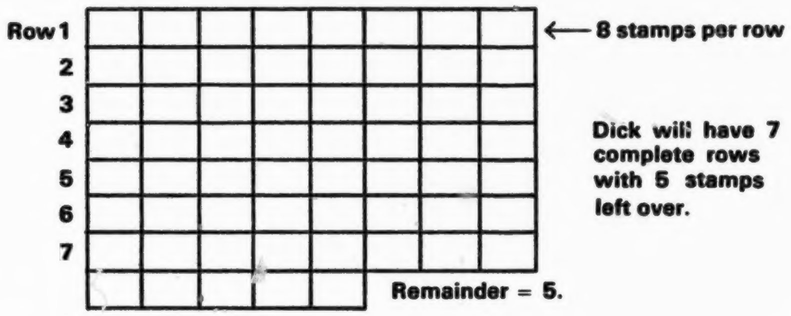
$$61 = (q \times 8) + r \quad \text{and} \quad r < 8.$$

To fulfill the requirements of the problem, we must find a quotient q

and a remainder r such that the above mathematical sentence will be true. By *trial* we find that

$$61 = (7 \times 8) + 5.$$

The following array interprets this relationship in terms of stamps on a page of the stamp book:

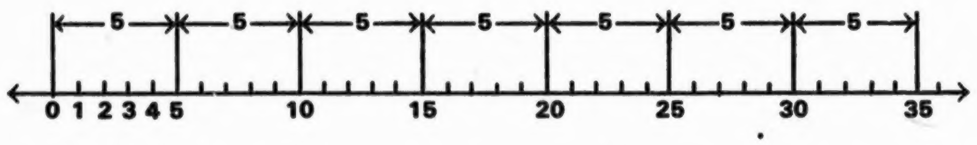

Row 1
2
3
4
5
6
7
8 stamps per row
Dick will have 7 complete rows with 5 stamps left over.
Remainder = 5.

Exercise Set 2

For the following mathematical sentences, determine the numbers that will make each sentence true, selecting for r the smallest number possible.

1. $75 = (q \times 6) + r.$
2. $120 = (q \times 12) + r.$
3. $11 = (q \times 7) + r.$
4. $6 = (q \times 8) + r.$
5. $15 = (q \times 15) + r.$

In Example 5 we gave a number-line interpretation of a simple division situation. We can also give a number-line interpretation to this extension of the idea of division. Referring to Example 5, let us assume that the senior class buys 1 yard (36 inches) of ribbon to make 5-inch badges. How many badges can be made, and how much ribbon will remain?


5
5
5
5
5
5
5
0 1 2 3 4 5
10
15
20
25
30
35

The diagram shows that seven badges can be made, with one inch of ribbon left over. This situation may be expressed as follows:

$$36 = (7 \times 5) + 1.$$

DIVISION WITH LARGE NUMBERS

Thus far we have been working with small numbers, for which our knowledge of multiplication "facts" enabled us to determine a solution by inspection. Let us consider a mathematical sentence involving larger numbers, for example:

$$138 = (q \times 17) + r.$$

We could use repeated subtraction, as we did before; that is, we could subtract the divisor repeatedly. But, instead, let us work with multiples of the divisor. Let us try to rename 138 as a sum of a multiple of 17 and a remainder. Let's make a guess and use 5 x 17, or 85, as our multiple of 17. Then we have

$$\begin{aligned} 138 &= 85 + 53 \\ &= (5 \times 17) + 53. \end{aligned}$$

But $53 > 17$, so let us rename 53 as a sum of two addends with at least one of the addends a multiple of the divisor (17). Since $5 \times 17 > 53$, let us try 3 x 17, or 51. Then $53 = (3 \times 17) + 2$, and we now have

$$138 = (5 \times 17) + [(3 \times 17) + 2].$$

Using the associative property of addition, we have . . $= [(5 \times 17) + (3 \times 17)] + 2.$

Using the distributive property, we have . . . . $= [(5 + 3) \times 17] + 2.$

Computing the sum of 5 and 3, we have. . . . . $= [(8) \times 17] + 2.$

Therefore $138 = (8 \times 17) + 2.$

Thus, we have determined that $q = 8, \; r = 2.$

Notice that the process terminates because our remainder, 2, is less than the divisor, 17. If our remainder had been greater than 17, the process could have been continued until the remainder became less than the divisor. To deal with larger numbers, we develop algorithms that will simplify the work for the student.

First steps in leading students to develop an efficient algorithm can begin with the method discussed below:

EXAMPLE 9: Jack is helping his father get eggs ready to take to market. His task is to place the eggs in cartons; each carton holds 12 eggs. If he has 159 eggs to box, how many cartons will he have and how many eggs will be left over?

The relationship for this problem may be expressed by

$$159 = (q \times 12) + r \quad \text{and} \quad r < 12.$$

Remember that we are trying to determine two numbers q and r that will meet the following requirements:

1. $159 = (q \times 12) + r$.
2. $r < 12$.

Let us use an organized way of estimating the answer:

$$\begin{array}{rcl} 1 \times 12 = 12; & & 12 < 159. \\ 10 \times 12 = 120; & & 120 < 159. \\ 100 \times 12 = 1200; & \text{but} & 1200 > 159. \end{array}$$

Therefore our quotient, q, is in the interval between 10 and 100, and is closer to 10 than to 100. So let us check the following products:

$$\begin{array}{rl} 11 \times 12 = 132; & 132 < 159. \\ 12 \times 12 = 144; & 144 < 159. \\ 13 \times 12 = 156; & 156 < 159. \end{array}$$

We can see that $(159 - 156) < 12$; therefore, the mathematical sentence for our problem may be written as

$$159 = (13 \times 12) + 3.$$

Answer: Jack will have 13 cartons with 3 eggs left over.

REFINING THE ALGORITHM

Now that we have developed the necessary background, a next step in refining our algorithm might consist of the following procedure: Recall the repeated-subtraction process used at the beginning of this chapter. We shall use that process as the basic idea; but, instead of subtracting the divisor once for each step, we shall subtract multiples of the divisor.

EXAMPLE 10: $138 = (q \times 7) + r$.

```
7)138 |
      |
   35 |  5     (5 x 7 = 35)
  ----|
  103 |
      |
```

Teacher: How many sevens do you think we can subtract from 138?

Sue: 5.

Bob: 10.

Don: 11.

Teacher: We could try all of these suggestions. Let's try 5 first.† After subtracting the 5 sevens, we have 103 left. Can we subtract any sevens from 103?

---

† Of course, when the suggestions made by Bob and Don are followed, the children will note that there is less written work. Later, children will be encouraged to try multiples of 10 (and powers of 10) as partial quotients.

EXAMPLE 10—*Continued*

[103]		
70	10	(10 x 7 = 70)
33		
28	4	(4 x 7 = 28)
5	19	
5	19	

Jim: I think we could use a number larger than 5. Let's try 10.

Teacher: Now we have 33 left. Can we subtract any sevens from 33?

Joe: Not as many as 5, because 5 sevens is 35. So let's try 4 sevens.

Teacher: We now have 5 left. Can we subtract any sevens from 5?

Sally: No. So that is our remainder.

Teacher: If 5 is our remainder, what is our quotient?

Dick: We subtracted 5 sevens, and 10 sevens, and 4 sevens. Altogether we subtracted 19 sevens. Our quotient is 19, and our remainder is 5.

In Example 10 the results were as follows: $q = 19$, $r = 5$.

The mathematical sentence for the relationship is $138 = (19 \times 7) + 5$.

Exercise Set 3

1. A teacher gave a problem to her class. Shown below is the work done by three of her students. Which solution is correct? Why?

Tom's solution	
9)781	
72	8
709	
180	20
529	
270	30
259	
225	25
34	
27	3
7	86

Carol's solution	
9)781	
27	3
754	
90	10
664	
450	50
214	
90	10
124	
45	5
79	
72	8
7	86

Dave's solution	
9)781	
45	5
736	
45	5
691	
108	12
583	
108	12
475	
108	12
367	
108	12
259	
108	12
151	
108	12
43	
36	4
7	86

2. Using the method of repeated subtraction, determine the q and r that will make the following sentences true, with r as small as possible:

- **a.** $685 = (q \times 13) + r$.
- **b.** $1{,}298 = (q \times 48) + r$.
- **c.** $7{,}592 = (q \times 73) + r$.
- **d.** $2{,}963 = (q \times 85) + r$.

A further step in developing an algorithm will be shown in Example 11:

EXAMPLE 11: The members of the senior class are making plans for selling tickets to their play. Each person is responsible for selling 8 tickets. If they take 1,900 tickets to be packaged in bundles of 8, how many bundles will they have, and how many tickets will be left over?

The mathematical sentence expressing the relationship in this problem is $1{,}900 = (q \times 8) + r$ and $r < 8$.

Let us use a form of estimation in which we work with multiples of powers of ten (that is, multiples of 1, of 10, of 100, and so on).

$8\overline{)1{,}900}$		1. Determine the interval:
		$1 \times 8 = 8$; $8 < 1{,}900$.
		$10 \times 8 = 80$; $80 < 1{,}900$.
		$100 \times 8 = 800$; $800 < 1{,}900$.
		$1{,}000 \times 8 = 8{,}000$; but $8{,}000 > 1{,}900$.
		Therefore our quotient is between 100 and 1,000.
		2. Determine the largest multiple of 100 that we can subtract.
		$100 \times 8 = 800$; $800 < 1{,}900$.
		$200 \times 8 = 1{,}600$; $1{,}600 < 1{,}900$.
		$300 \times 8 = 2{,}400$; but $2{,}400 > 1{,}900$.
$\underline{1{,}600}$	200	Therefore we subtract 200×8, or 1,600.
300		3. Determine the largest multiple of 10 that we can subtract.
		$10 \times 8 = 80$; $80 < 300$.
		$20 \times 8 = 160$; $160 < 300$.
		$30 \times 8 = 240$; $240 < 300$.
		$40 \times 8 = 320$; but $320 > 300$.
$\underline{240}$	30	Therefore, we subtract 30×8, or 240.
60		4. By inspection, we should be able
$\underline{56}$	7	to determine that we can subtract
$r = 4$	$237 = q$	7×8, or 56.

From these results, we may state that

$$1{,}900 = (237 \times 8) + 4.$$

Interpretation of the results: 237 bundles of 8 tickets each, with 4 tickets left over.

Exercise Set 4

Using the technique outlined in the above problem, determine the q and r that will make the following sentences true, with r as small as possible.

1. $187 = (q \times 12) + r.$

2. $868 = (q \times 25) + r.$

3. $2{,}448 = (q \times 39) + r.$

4. $2{,}205 = (q \times 21) + r.$

One of the major difficulties in working with most division algorithms is that of determining at each step the largest number that can be used as a partial quotient. Before using another technique, let us review a principle that you would want to develop with your students.

EXAMPLE 12:

(1) $24 \div 6 = \square.$

If we divide both 24 and 6 by 2, we obtain

(2) $12 \div 3 = \square.$

How do the answers to (1) and (2) compare?

EXAMPLE 13:

(1) $84 \div 12 = \square.$

Divide both 84 and 12 by 2.

(2) $42 \div 6 = \square.$

Divide both 84 and 12 by 4.

(3) $21 \div 3 = \square.$

Divide both 84 and 12 by 6.

(4) $14 \div 2 = \square.$

Divide both 84 and 12 by 12.

(5) $7 \div 1 = \square$.

Divide both 84 and 12 by 3.

(6) $28 \div 4 = \square$.

Compare the answers to (1), (2), (3), (4), (5), and (6). What do you observe about these answers?

As a result of working with these and similar examples, your students should be able to formulate the following generalization:

> ***Dividing both the dividend and the divisor by the same number has no effect on the quotient.***

As a next step in refining a division algorithm, consider the following examples:

EXAMPLE 14:

$$\begin{array}{r|l} 6\overline{)257} & \\ 240 & 40 \\ \hline 17 & \\ 12 & 2 \\ \hline r = 5 & 42 = q \end{array}$$

1. Determine the interval:
$$\begin{array}{rl} 1 \times 6 = 6; & 6 < 257. \\ 10 \times 6 = 60; & 60 < 257. \\ 100 \times 6 = 600; & 600 > 257. \end{array}$$
Therefore, our partial quotient is between 10 and 100.
2. To make the estimation easier, we round 257 to the nearest multiple of 10. We try
$$6\overline{)260}$$
3. By inspection, we determine that our partial quotient is 40.
4. A student's knowledge of multiplication facts should enable him to see that the next partial quotient is 2.

EXAMPLE 15:

$$\begin{array}{r|l} 18\overline{)856} & \\ & \end{array}$$

1. Determine the interval:
$$\begin{array}{rl} 1 \times 8 = 18; & 18 < 856. \\ 10 \times 18 = 180; & 180 < 856. \\ 100 \times 18 = 1{,}800; & 1{,}800 > 856. \end{array}$$
Therefore, our quotient is between 10 and 100.
2. For estimation purposes:
   a) Round the divisor and the dividend to the nearest multiple of 10. We try
   $$20\overline{)860}.$$

EXAMPLE 15—*Continued*

|856|	
720	40
136	
126	7
$r =$ 10	47 $= q$

b) In light of the discussion in Examples 12 and 13, we see that (by dividing dividend and divisor by 10) we can rewrite the above as

$$2\,\overline{)86.}$$

c) By inspection, we determine that our partial quotient is about 40.

3. For further estimation:
   a) Round the divisor and dividend to the nearest multiple of 10. Try

$$20\,\overline{)140.}$$

   b) This quotient is equivalent to

$$2\,\overline{)14.}$$

   c) By inspection, we determine that our partial quotient is 7.

EXAMPLE 16:

$12\overline{)236}$	
120	10
116	
108	9
$r =$ 8	19 $= q$

1. Determine the interval:

$1 \times 12 = 12;\quad 12 < 236.$
$10 \times 12 = 120;\quad 120 < 236.$
$100 \times 12 = 1{,}200;\quad 1{,}200 > 236.$

2. Think of the problem as

$$10\overline{)240,}$$

then as

$$1\overline{)24.}$$

It would appear that the partial quotient is 20, but

$20 \times 12 = 240$ and $240 > 236.$

So we must use 10.

3. By inspection, we determine that the partial quotient is 9.

NOTE: We see by this example that our estimation techniques are not always effective—this is why division is difficult. Although these techniques for determining partial quotients are not "perfect," they are satisfactory for the majority of problems. If the technique fails at any stage, we have a way of estimating a new partial quotient.

Observe, however, that if we round *up* the divisor and round *down* the dividend, the estimated quotient will never be too large.

We hope that students eventually will be able to solve problems of this type with a minimum of writing. The following example indicates how this might be done.

EXAMPLE 17:

$18\overline{)4,262}$		1. To determine a partial quotient, round the numbers and think of the example as $$20\overline{)4,300}$$ or as $$20\overline{)43\text{ hundreds}}.$$
$\underline{3,600}$ 662	200	Our estimate would be 2 hundreds, or 200.
		2. Round the numbers and think of the example as $$20\overline{)660}$$ or as $$20\overline{)66\text{ tens}}.$$
$\underline{540}$ 122	30	Our estimate would be 3 tens, or 30.
		3. Round the numbers and think of the example as $$20\overline{)120}$$ or as $$2\overline{)12}.$$
$\underline{108}$ $r =$ 14	6 236 $= q$	Our partial quotient is 6.

Let's look at another example.

EXAMPLE 18:

$24\overline{)6,080}$		1. To determine a partial quotient, think of the example as $$20\overline{)6,100}$$ or as $$20\overline{)61\text{ hundreds}}.$$ Estimation: 3 hundreds. But $$300 \times 24 = 7,200$$ and $$7,200 > 6,080.$$
$\underline{4,800}$ 1,280	200	Therefore, our partial quotient is 200, or 2 hundreds.

EXAMPLE 18—*Continued*

$$\begin{array}{r|l} [1{,}280] & \\ \underline{1{,}200} & 50 \\ 80 & \\ \underline{72} & \underline{3} \\ r = 8 & 253 = q \end{array}$$

2. Think of the example as
$$20\overline{)1{,}300}$$
or as
$$20\overline{)130 \text{ tens.}}$$
Estimation: 6 tens, or 60.
But
$$60 \times 24 = 1{,}440$$
and
$$1{,}440 > 1{,}280.$$
Therefore, our partial quotient is 5 tens, or 50.
3. By inspection, we can see that our partial quotient is 3.

In the next example, let us write our quotient in a different position; this form is more like that of the conventional algorithm.

EXAMPLE 19:

$$\begin{array}{r l} \underline{207} & \leftarrow q \\ \underline{7} & \\ 200 & \\ 48\overline{)9{,}974} & \\ \underline{9{,}600} & \\ 374 & \\ \underline{336} & \\ 38 & \leftarrow r \end{array}$$

Think:
1. $50\overline{)100 \text{ hundreds}}$ Partial quotient: 2 hundreds.
2. $50\overline{)37 \text{ tens}}$ Partial quotient: 0 tens.
3. $50\overline{)370}$ or $5\overline{)37}$ Partial quotient: 7.

Answer: $q = 207$, $r = 38$.

Example 18, written in the above form, would look like this:

$$\begin{array}{r l} \underline{253} & \leftarrow q \\ 3 & \\ 50 & \\ 200 & \\ 24\overline{)6{,}080} & \\ \underline{4{,}800} & \\ 1{,}280 & \\ \underline{1{,}200} & \\ 80 & \\ \underline{72} & \\ 8 & \leftarrow r \end{array}$$

Answer: $q = 253$, $r = 8$.

Exercise Set 5

Using one of the methods outlined above, compute:

1. $9\overline{)5{,}268}$
2. $27\overline{)4{,}693}$
3. $52\overline{)10{,}000}$
4. $76\overline{)30{,}421}$

In the case of students who have difficulty with division, it would probably be best to let them continue to use one of the forms illustrated in Examples 11, 15, and 17—the choice depending on the level at which they can work most effectively.

But with other students you would want to use the conventional algorithm. Let us look at an example in which the conventional form is used.

EXAMPLE 20:

$$42\overline{)6{,}893}$$

Think:

1. $40\overline{)69 \text{ hundreds}}$ Partial quotient: 1 hundred.

$$\begin{array}{r} 1\\ 42\overline{)6{,}893}\\ \underline{4{,}200}\\ 2{,}693 \end{array}$$

Instead of writing the partial quotient as "100," we write only "1" in the position immediately above the hundreds position in the dividend.

2. $40\overline{)270 \text{ tens}}$ Partial quotient: 6 tens. Instead of writing the partial quotient as "60," we write only "6" in the position above the tens position in the dividend.

$$\begin{array}{r} 16\\ 42\overline{)6{,}893}\\ \underline{4{,}200}\\ 2{,}693\\ \underline{2{,}520}\\ 173 \end{array}$$

3. $40\overline{)170}$ Partial quotient: 4. We write "4" in the position immediately above the ones position in the dividend.

$$\begin{array}{r} 164\\ 42\overline{)6{,}893}\\ \underline{4{,}200}\\ 2{,}693\\ \underline{2{,}520}\\ 173\\ \underline{168}\\ 5 \end{array}$$

Answer: $q = 164$, $r = 5$.

SUMMARY

In this chapter we have discussed several approaches to the solution of two basic types of problems—problems that are division situations and problems that are an extension of the division idea. The approach best suited to a student depends, of course, upon his aptitude for mathematics, general ability, and mathematical background. A child begins at the physical level, working with sets of objects and partitioning them into equivalent subsets. He then progresses to an algorithm based upon the activity he carried out at the physical level—instead of "subtracting off" equivalent subsets of objects, he now repeatedly subtracts the divisor. It is important that the student understand this algorithm, since most other algorithms are modifications of this one.

These modifications consist basically of subtracting a few multiples of the divisor instead of subtracting the divisor itself a large number of times. Later we introduce further modifications aimed at developing efficient techniques for determining the largest *multiples* of the divisor that can be subtracted, so that the least number of subtractions will be needed. The goal for most students is the conventional algorithm, for which there is a minimum of writing. However, other algorithms lead fairly quickly to the correct answer and are basically satisfactory.

While working with division situations, we found instances that were similar to division but for which we could not use the "missing factor" concept. For instance, in Example 1, we were able to express the relationship as

$$12 = \boxed{4} \times 3.$$

(This corresponds to the case where the remainder is zero.) But when we changed the data so that the teacher had 13 chocolates, there was no whole number that could make the following relationship true:

$$13 = \square \times 3.$$

Because of situations of this type, it was necessary to extend the idea of division and to develop the following generalization:

For any pair of whole numbers a and b, where $b \neq 0$, it is always possible to determine a pair of whole numbers q (quotient) and r (remainder) such that

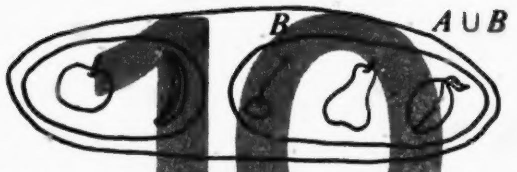
$$a = (q \times b) + r, \quad \text{and} \quad r < b.$$

a: dividend
b: divisor
q: quotient
r: remainder

With this generalization, we can then describe the situation involving the 13 chocolates as

$$13 = (4 \times 3) + 1 \quad \text{where} \quad 4 \text{ [illegible] } q \quad \text{and} \quad 1 = r.$$

This generalization can be used not only for situations that are an extension of the idea of division but also for division itself, which corresponds to the case where the remainder is zero. The remainder will be zero if and only if the divisor and the quotient are factors of the dividend.



10

THE WHOLE-NUMBER SYSTEM —KEY IDEAS

1. **What are some of the key ideas of the whole-number system that permeate much of mathematics?**
2. **How are sets used in developing the whole-number system?**
3. **Why is renaming an important notion in elementary mathematics?**
4. **What are the differences in meaning between the terms "operation," "computation," and "algorithm"?**
5. **Why is it desirable to start the notion of proof in elementary mathematics?**

As elementary school teachers, many of us probably feel that the main contribution elementary mathematics makes to the education of our children is to enable them to solve the problems of a numerical or logical nature that they are most likely to meet in everyday life. Another important aim of elementary mathematics should be to provide children with a good basic foundation for later mathematics courses, which many will need in preparing for their careers. In order to accomplish these objectives in a minimum amount of time, we seek ideas that simplify, clarify, and unify our thinking. These ideas, which we are calling KEY IDEAS, will be our main concern here.

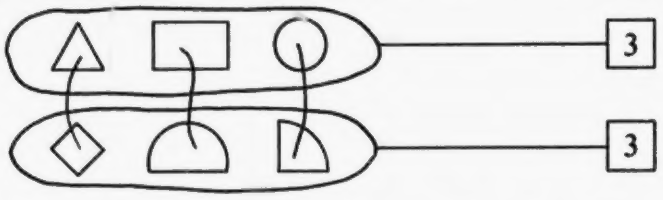
The key ideas we have considered in developing the whole-number system permeate many areas of mathematics, so that when we devote class time to these ideas, we prepare children to meet not only their immediate mathematical needs but also the mathematical needs of their future.

One of the key ideas permeating much of mathematics is the concept of a set.

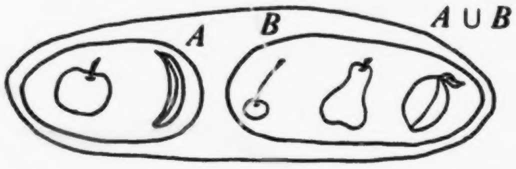
SET

In our approach to the whole-number system, we used sets in many ways.

1. Through *equivalent* sets we conveyed the notion of a whole number.



2. Through the union of *disjoint* sets we defined the sum of a pair of whole numbers.

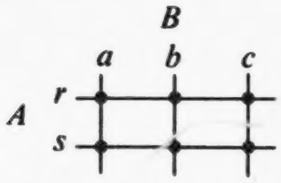


Hence $n(A) = 2.\ n(B) = 3.$

$$\begin{aligned} 2 + 3 &= n(A) + n(B) \\ &= n(A \cup B) \\ &= 5. \end{aligned}$$

$$n(A) + n(B) = n(A \cup B).$$

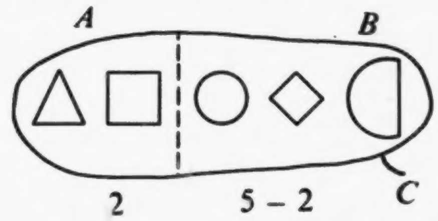
3. Through the *cross product* of a pair of sets we showed how the product of a pair of whole numbers can be defined.

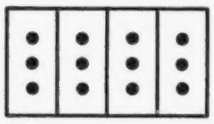


$$\begin{aligned} n(A) &= 2.\ n(B) = 3. \\ \text{Hence}\quad 2 \times 3 &= n(A) \times n(B) \\ &= n(A \times B) \\ &= n(\{(r, a), (r, b), (r, c), \\ &\qquad (s, a), (s, b), (s, c)\}) \\ &= 6. \end{aligned}$$

$$n(A) \times n(B) = n(A \times B)$$

4. Through a *partitioning* of sets we showed how the concepts of difference and quotient may be developed.





$12 \div 3 = 4$

or

$12 \div 4 = 3.$

In some of the newer programs children are taught to translate problems into mathematical sentences. They also learn to solve mathematical sentences and are introduced to the important notion of a solution set. Here are some typical mathematical sentences and the solution set for each.

Find the set of whole numbers satisfying these conditions:

Sentence	*Solution Set*
$2 \times \square = 8.$	$\{4\}$
$2 \times \square < 8.$	$\{0, 1, 2, 3\}$
$2 \times \square = 9.$	$\{\ \}$
$\square$ is even and less than 8.	$\{0, 2, 4, 6\}$

Exercise Set 1

1. Which pairs of these sets are equivalent?

$A = \{a\}$. $C = \{\text{John, Mary}\}$. $E = \{7\}$. $G = \{1, \text{blue}\}$.

$B = \{2\}$. $D = \{0, 1, 2\}$. $F = \{a, b, c\}$. $H = \{\ \}$.

2. What requirement determines whether two sets are equivalent?

3. What number would you associate with each of the sets mentioned in Exercise 1? [For example: $n(A) = 1$.]

4. When are two sets assigned the same number?

5. Which pairs of the sets mentioned in Exercise 1 are *not* disjoint?

6. When are two sets disjoint?

7. With the use of sets, how can one compute the sum $2 + 3$?

8. If a and b are whole numbers, define $a + b$ in terms of sets.

9. Express the following cross products of sets mentioned in Exercise 1 by listing the members within braces. For example: $B \times C = \{(2, \text{John}), (2, \text{Mary})\}$.

a. $A \times B$

b. $A \times C$

c. $C \times D$

d. $D \times C$

e. $D \times E$

f. $D \times F$

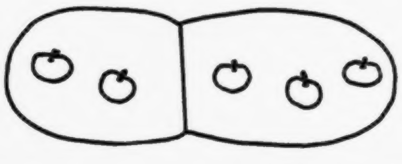
g. $D \times G$

h. $D \times H$

10. Find the number of elements (ordered pairs are the elements here) in each cross product mentioned in Exercise 9. What related mathematical sentence is suggested by each cross product?

11. If a and b are whole numbers, explain how $a \times b$ can be defined in terms of the cross product of two sets.

12.



a. What addition sentence is suggested by the above figure?

b. What subtraction sentences are suggested by the above figure?

13. If the whole number a is not less than the whole number b, define $a - b$ in terms of a partitioning of a set.

14. Show how $12 \div 3$ may be illustrated by two different ways of partitioning a set of 12 elements.

15. Find a set of whole numbers satisfying each of these conditions:

a. $3 + \square = 7.$

b. $\square - 3 = 7.$

c. $3 \times \square = 7.$

d. $3 \times \square = 6.$

e. $3 + \square < 7.$

f. $\square - 3 < 7.$

g. $3 \times \square < 7.$

h. $\square \div 3 < 7.$

i. $\square$ is even and
$\square \div 3 < 7.$

PROPERTIES OF OPERATIONS

In order that our pupils learn to compute efficiently, we teach them algorithms. But we want children to understand the reasoning behind each step of an algorithm, so we devote time to the properties of our four fundamental arithmetic operations. With a meaningful interpretation of the operations and their properties, children are in a position to figure out many of the algorithms or to compute without the need of knowing any prescribed algorithm. Of prime importance here is the role played by our decimal numeration system, which is used internationally and in all mathematics.

The main operation properties treated in these units may be summarized as follows:

Addition and multiplication are each commutative.

$$a + b = b + a; \quad a \times b = b \times a.$$

Addition and multiplication are each associative.

$$(a + b) + c = a + (b + c); \quad (a \times b) \times c = a \times (b \times c).$$

Multiplication distributes with respect to addition:

$$a \times (b + c) = (a \times b) + (a \times c).$$

For addition the identity element is 0, while for multiplication the identity element is 1.

$$\begin{array}{lcl} 0 + a = a & \text{and} & a + 0 = a. \\ 1 \times a = a & \text{and} & a \times 1 = a. \end{array}$$

The roles of 0 and 1 in subtraction and division may be summarized as follows: For every whole number a—

$$\begin{array}{l} a - 0 = a. \\ a - a = 0. \\ 0 \div a = 0 \quad (\text{for } a \neq 0). \\ a \div a = 1 \quad (\text{for } a \neq 0). \\ a \div 1 = a. \end{array}$$

RENAMING

The properties mentioned above may be viewed as "renaming" properties. For example, the number named by "283 + 794" is the same as the number named by "794 + 283," and we need not compute the sum to know this.

$$283 + 794 = 794 + 283.$$

The equality follows from the commutative property of addition.

Renaming is a basic idea in mathematics. We are continually renaming numbers in arithmetic computation. In fact, computation is a renaming process, since it requires that we find a standard name for a number.

REASONING AND PROOF

Although we have not exactly spelled out a formal development of the whole-number system, we have approximated what is referred to as a "postulational" approach. Although we have reasoned from

(1) undefined terms—for example, "set"
(2) defined terms—for example, "$a + b$"
(3) assumptions—for example, that $(a \times b) \times c = a \times (b \times c)$,

we have not given actual formal proofs of our properties. We have, hopefully, made them reasonable. Although the operation properties in our development have been, for the most part, assumptions, proofs for these properties can be found in more advanced treatments.

Nevertheless, the notion of proof is a key idea in mathematics. Although in the early grades we do not deal with proof in a formal way, a beginning is made when we ask, "How do you know that 2 + 3 = 5?" and the child holds up 2 fingers and 3 more fingers and then counts to 5. A further development takes place when children justify mathematical statements by citing properties of the operations rather than by resorting to computation. Thus suppose we have to compute

$$(1) \qquad \begin{array}{r} 25 \\ \underline{\times 476} \end{array}$$

Because multiplication is commutative, we know that

$$25 \times 476 = 476 \times 25.$$

Consequently, the computation in (1) may be replaced by

$$(2) \qquad \begin{array}{r} 476 \\ \underline{\times \ \ 25} \end{array}$$

which is somewhat shorter.

Another stage in developing the notion of proof is illustrated by the following argument involving even and odd numbers. We begin with certain definitions and properties.

THE DEFINITIONS:

An even number is the double of some whole number. It may be represented by $2a$ where a stands for some whole number. These are even numbers: $2 \times 0, 2 \times 1, 2 \times 2, 2 \times 3, 2 \times 4, \ldots$.

An odd number is 1 more than some even number. It may be represented by $2a + 1$ where a stands for some whole number. These are odd numbers: $(2 \times 0) + 1, (2 \times 1) + 1, (2 \times 2) + 1, (2 \times 3) + 1, \ldots$.

THE PROPERTIES:

The sum of a whole number and a whole number is a whole number. (This follows from our definition of sum.)

Multiplication is distributive over addition. (We call this property "distributivity" for short.)

Using these definitions and properties, we can proceed in the following manner to give a formal proof that the sum of two even numbers is an even number.

Represent the first even number by $2a$ and the second even number by $2b$, where a and b are whole numbers, not necessarily different. Then

$2a + 2b = 2(a + b)$	by the distributive property;
$a + b$ is a whole number	because the sum of whole numbers is a whole number.
Hence $2(a + b)$ is even	by our definition of an even number, for it is twice a whole number.

This argument proves that the sum of an even number and an even number is an even number. (We may abbreviate this generalization by writing "E + E = E.") In a similar manner we can prove that—

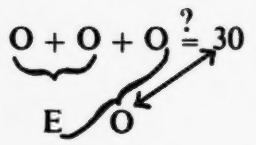
(1) The sum of an odd number and an odd number is an even number. (We may abbreviate this generalization by writing "O + O = E.")

(2) The sum of an even number and an odd number is an odd number. (We may abbreviate this generalization by writing "E + O = O" or "O + E = O.")

On the basis of these generalizations about even and odd numbers, one might challenge brighter pupils to prove that there are no three odd numbers totaling 30. (The three odd numbers need not be different.) The proof is based on the generalizations that

$$O + O = E \qquad \text{and} \qquad E + O = O$$

and on the fact that 30 is an even number, because $30 = 2 \times 15$. Given three odd numbers (not necessarily different), the sum of the first two odd numbers is an even number. The sum of this even number and the third odd number is an odd number. But 30 is an even number. So we conclude that there cannot be three odd numbers that have a sum of 30. The reasoning may be shown schematically as follows:



These generalizations about odd and even numbers can serve as an additional check on computation. Thus, one can say that the following computed sum is incorrect because E + E = E—that is, the sum of a pair of even numbers is an even number—whereas 65 is odd.

$$\begin{array}{r} 26 \\ +\ 38 \\ \hline 65 \end{array} \quad \text{Incorrect}$$

Exercise Set 2

NOTE.—Exercises below marked "☆" take us into the realm of algebra. They are *not* being suggested for use in elementary school except possibly for special work with advanced groups. They are offered as exercises in deductive reasoning for teachers who wish to explore, in greater depth, proofs for steps in algorithms.

1. What properties or definitions are used in the following?

a. $39 + 76 = 76 + 39$.

b. $(39 + 76) + 24 = 39 + (76 + 24)$.

c. $\begin{array}{r} 35 \\ +\ 2 \\ \hline 37 \end{array}$ $\left[\begin{array}{l} \text{Hint: } 35 + 2 = (30 + 5) + 2 \\ \qquad\qquad\quad\;\; = 30 + (5 + 2) \end{array}\right]$

d. $\begin{array}{r} {}^{1} \\ 35 \\ +\ 9 \\ \hline 44 \end{array}$

e. $\begin{array}{r} 35 \\ -\ 2 \\ \hline 33 \end{array}$

f. $\begin{array}{r} {}^{2}\;{}^{15} \\ \cancel{3}\cancel{5} \\ -\ 9 \\ \hline 2\;6 \end{array}$

g. $35 - 9 = 36 - 10$.

h. $\begin{array}{r} 32 \\ \times\ 3 \\ \hline 96 \end{array}$

i. $\begin{array}{r} 34 \\ \times\ 3 \\ \hline 102 \end{array}$

j. $37 \times 25 = 25 \times 37$.

k. $(37 \times 25) \times 4 = 37 \times (25 \times 4)$.

l. $(37 \times 25) \times 0 = 0$.

m. $(37 \times 73) + (23 \times 73) = (37 + 23) \times 73$.

n. $12 \div 3 = 4$.

o. $12 \div 1 = 12$.

p. "$12 \div 0$" and "$0 \div 0$" are meaningless expressions.

2. Find the appropriate number for each frame:

a. $678 = (\square \times 100) + (\triangle \times 10) + (\triangledown \times 1)$.

b. $\square + 38 = 78 - \square$.

(Remember to use the same number for each frame of the same shape.)

3. Justify: If a and b are any whole numbers, then

$$(a + b) - b = a.$$

For example: $(4 + 3) - 3 = 4$.

☆ **4.** Justify: If a and b are any whole numbers with a not less than b, then

$$(a - b) + b = a.$$

For example: $(4 - 3) + 3 = 4$.

☆ **5.** Justify: For any whole numbers a, b, and c with b not less than c,

$$(a + b) - c = a + (b - c).$$

For example: $(4 + 3) - 2 = 4 + (3 - 2)$.

☆ **6.** Justify: For any whole numbers a, b, c with a not less than $b + c$,

$$(a - b) - c = a - (b + c).$$

For example: $(7 - 3) - 1 = 7 - (3 + 1)$.

☆ **7.** Justify: For any whole numbers a, b, c with a not less than b,

$$(a - b) = (a + c) - (b + c).$$

For example: $15 - 7 = (15 + 3) - (7 + 3)$.

☆ **8.** Justify: For any whole numbers a, b, c with a not less than b and b not less than c,

$$(a - c) - (b - c) = a - b.$$

For example: $(15 - 5) - (7 - 5) = 15 - 7$.

☆ **9.** Justify: For any whole numbers a, b, c, d with a not less than c, and b not less than d,

$$(a + b) - (c + d) = (a - c) + (b - d).$$

For example: $(20 + 7) - (10 + 3) = (20 - 10) + (7 - 3)$.

☆ **10.** Justify: For any whole numbers a, c, d with c not less than d, and a not less than c,

$$a - (c - d) = (a - c) + d.$$

For example: $17 - (7 - 3) = (17 - 7) + 3$.

☆ **11.** Justify: For any whole numbers a, b, and c with a not less than b,

$$(a - b) + c = (a + c) - b.$$

For example: $(17 - 2) + 3 = (17 + 3) - 2$.

☆ **12.** Justify: For any whole numbers a, b, and c with b not less than c,

$$a + b = (a + c) + (b - c).$$

For example: $7 + 12 = (7 + 2) + (12 - 2)$.

Corresponding to each of the generalizations in Exercises 3 through 12 there is a generalization involving multiplication and division, multiplication corresponding to addition and division corresponding to subtraction. Thus, corresponding to

$$(a + b) - b = a$$

we have

$$(a \times b) \div b = a.$$

Moreover, the proofs are essentially analogous.

☆ **13.** Justify: For any whole numbers a and b, with $b \neq 0$,
(1) $(a \times b) \div b = a$.
(2) $(a \div b) \times b = a$, provided $a \div b$ is a whole number.

☆ **14.** State generalizations involving multiplication and division corresponding to those in Exercises 4 through 11 and try to justify each.

☆ **15.** In the definition of $a + b$, show that it does not matter which sets A and B are used just as long as $n(A) = a$, $n(B) = b$, and A and B are disjoint. (In our definition we assumed that it did not matter. Here, we are seeking to show that this assumption can be established.)

16. Find a short way to compute the following:
a. $(36 \times 47) + (47 \times 64)$

b. $25 \times (36 \times 4)$

c. $575 - 298$

d. $575 + 298$

e. $575 \div 25$

☆ **17.** Justify: If $a < b$, then
(1) $a + c < b + c$;

(2) $a - c < b - c$ provided a is not less than c;

(3) $c - a > c - b$ provided c is not less than b;

(4) $a \times c < b \times c$ provided $c \neq 0$.

☆ **18.** Justify: If $a < b$ and $c < d$, then

(1) $a + c < b + d$;

(2) $a \times c < b \times d$.

☆ **19.** Justify: If $a < b$ and $b < c$, then $a < c$.

☆ **20.** Justify:

a. The sum of two odd numbers is an even number.

b. The sum of an even number and an odd number is an odd number.

c. The product of two whole numbers is an even number whenever one of the factors is even.

d. The product of two odd numbers is an odd number.

21. Prove as you would expect a primary-grade child to prove:

a. $6 + 2 = 8$.

b. $6 - 2 = 4$.

c. $6 \times 2 = 12$.

d. $6 \div 2 = 3$.

CORRESPONDENCE

We turn next to another key idea that permeates much of mathematics, the notion of correspondence. In studying the whole-number system we touched on this idea when we considered one-to-one correspondence and the notion of an operation. Later on in mathematics, this idea is again introduced under the notion of a function.

Returning to the operations we have studied, let us note that each of the four basic operations of arithmetic associates a single number with a given pair of numbers.

	+	−	×	÷
NUMBER PAIR	SUM	DIFFERENCE	PRODUCT	QUOTIENT
(12, 3)	12 + 3, or 15	12 − 3, or 9	12 × 3, or 36	12 ÷ 3, or 4
(a, b)	$a + b$	$a - b$	$a \times b$	$a \div b$
		provided a is not less than b		provided there is a whole number c such that $a = b \times c$ and $b \neq 0$.

Because a pair of numbers must be specified first, the operations we have studied are often called *binary* operations. Addition, subtraction, multiplication, and division are not the only binary operations. In fact, one can show that there are an endless number of binary operations. Let's consider one other binary operation that most of us are familiar with, averaging a pair of numbers. For example, the average of 5 and 9 is

$$(5 + 9) \div 2 = 14 \div 2 = 7.$$

Suppose we let "v" denote the operation of averaging two numbers. We could then write

$$5 \text{ v } 9 = 7$$

or we could write

$$a \text{ v } b = \frac{a+b}{2}$$

when defined; that is, when a and b are either both even or both odd. (When rational numbers are studied, there is no such limitation.)

We could now ask, What properties does the operation of averaging enjoy? Is it commutative? Clearly, when $a \text{ v } b$ is defined, then so is $b \text{ v } a$; and $a \text{ v } b = b \text{ v } a$. This follows from the fact that $a + b = b + a$, since addition is commutative.

$$a \text{ v } b = \frac{a+b}{2} = \frac{b+a}{2} = b \text{ v } a.$$

Is averaging associative? Let's try some numbers.

$$(4 \text{ v } 8) \text{ v } 12 = 6 \text{ v } 12 = 9$$

while

$$4 \text{ v } (8 \text{ v } 12) = 4 \text{ v } 10 = 7.$$

But $9 \neq 7$. Hence,

$$(4 \text{ v } 8) \text{ v } 12 \neq 4 \text{ v } (8 \text{ v } 12).$$

This one exception is enough to *prove* that averaging is not associative.

In addition to binary operations, each of which associates some number with a given pair of numbers, there are also unary operations, which associate some number with a given single number. For example, doubling associates with every single number its double.

With 0, doubling associates 2 × 0, or 0.
" 1, " " 2 × 1, or 2.
" 2, " " 2 × 2, or 4.
" *a*, " " 2 × *a*, or 2*a*.

Number	Double
0	0
1	2
2	4
a	2*a*

Other unary operations are tripling, squaring, increasing by 7, and so on. In their later study of mathematics, children will meet the trigonometric functions and logarithmic functions, which are unary operations. In some of the newer programs a basis for such later developments is provided by introducing unary operations such as doubling, squaring, increasing by one, and so on.

Exercise Set 3

1. Fill in the table, wherever possible, with standard names for whole numbers.

NUMBER PAIR	+ SUM	− DIFFERENCE	× PRODUCT	÷ QUOTIENT	v AVERAGE
(12, 4)					
(44, 4)					
(4, 4).					
(4, 12)					
(2, 0).					
(0, 2).					
(0, 0).					

2. In the manner shown in the first row, fill in the table, wherever possible, with names for whole numbers.

NUMBER	DOUBLE	TRIPLE	SQUARE	SUCCESSOR	PREDECESSOR
6	2×6 or 12	3×6 or 18	6×6 or 36	$6 + 1$ or 7	$6 - 1$ or 5
2					
7					
1					
0					
10					

ORDER

Another idea that is playing an increasingly important role in mathematics is *order*. In our study of the whole numbers we touched on this notion when we worked with inequalities. In classical areas such as calculus, for example, and in more modern areas, such as linear programming, inequalities are indispensable.

Exercise Set 4

1. Using the addition operation, how can you show children that 5 is less than 7?

2. Using the addition operation, define what we mean when we say that a is less than b.

3. What restriction is needed for "$a - b$" to be meaningful in the study of whole numbers?

4. If $a < b$, show that $a + 1 < b + 1$.

5. Describe how a child can compare the number of pencils in a bag with the number of crayons in a box without counting.

6. If *a* and *b* are whole numbers, what are the three possibilities for expressing their relationship in terms of size?

SUMMARY

In our study of the whole-number system we have encountered some of the key ideas of mathematics:

1. The basic role *set* plays in mathematics
2. The importance of *renaming* in computation
3. The kind of *reasoning* we do in mathematics from undefined term, defined term, and assumption
4. The transition from *conjecturing* and plausible reasoning to the seeking of *proof* which will validate or invalidate these conjectures
5. *Correspondence* through the notions of one-to-one correspondence and operation. (An operation establishes a correspondence between two sets.)
6. *Order* in counting and in dealing with inequalities

With a firm grasp of these ideas, children will be better prepared both to figure out their own solutions to problems in elementary mathematics and to expand their knowledge of mathematics in the challenging years that lie ahead.

ANSWERS TO EXERCISES

ANSWERS TO EXERCISES

BEGINNING NUMBER CONCEPTS

Exercise Set 1, pp. 4-5

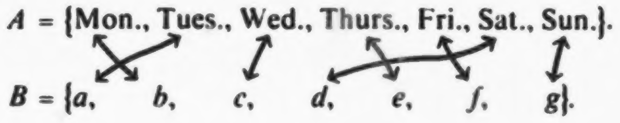
1. Place a finger on each dot. This sets up a pairing of fingers and dots. If each finger is paired with a particular dot so that each dot is paired with a particular finger, the sets are matched. If the two sets match, they are equivalent and have the same number of members.

2. Various correct answers are possible. Each person's name can be put on a separate piece of paper. Each paper is placed on a chair, one paper to a chair. If every paper with a name on it can be placed on a chair, then there are enough chairs.

3. There may be more chairs than members of the staff; there may be as many chairs as members of the staff; there may be fewer chairs than members of the staff.

4. $A = \{$Mon., Tues., Wed., Thurs., Fri., Sat., Sun.$\}$.

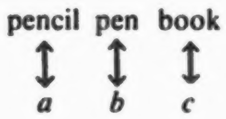
5. Here is one of many correct answers:

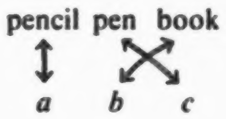

A = {Mon., Tues., Wed., Thurs., Fri., Sat., Sun.}.
B = {a, b, c, d, e, f, g}.

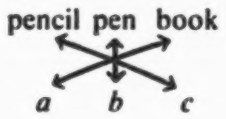
6. When the elements of one set have been paired with the elements of the other set so that the two sets match.

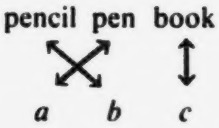
7. Sets A and E; sets B and D.

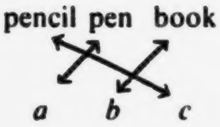
8.

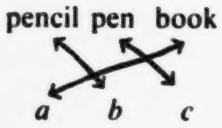












Exercise Set 2, pp. 7-8

1. If a one-to-one correspondence can be set up between two sets, they can belong to the same family.

2. They are equivalent; they have the same number of elements.

3. Seven.

4. Various correct answers are possible. One such set is the set of days of the week.

5. Various correct answers are possible.

E = {John, Henry, Bill, Mary, Jane, Jim, Susan}.

6. Sets D and E can belong to the same family of equivalent sets and therefore have the same number of elements.

7. Seven.

Exercise Set 3, pp. 10-11

1. All sets except E. A set S is a subset of set T if every member of set S is also a member of set T.

2. Sets A, B, C, D. To be a proper subset of G, every member of the subset must also be a member of set G, but there must be at least one member of G that is not a member of the subset. It is for this reason that set F is not a proper subset of set G.

3. Various correct answers are possible. For example, a proper subset of B is {black, pencil}.

{black, pencil}
↕ ↕
{Mary, John}

4. The number of set B is greater than the number of set A, or the number of set A is less than the number of set B.

5. 2 is less than 3.

6. The number of set A is less than the number of set B.

7. Various correct answers are possible. Since {hat} is a proper subset of set F and it can be matched with set E, $1 < 2$.

E = {chair}.
F = {hat, coat}.

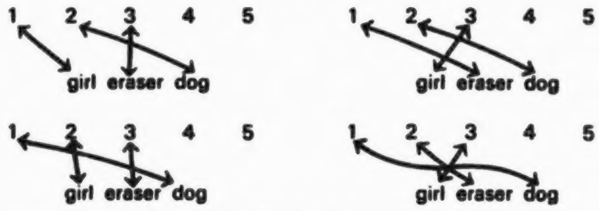
8. Various correct answers are possible:

R = {tree, cloud, hill}.
S = {John, pen, chain, house, window}.

There is a proper subset of set S which can be matched with set R. Therefore $n(R) < n(S)$.

Exercise Set 4, pp. 14-15

1.



2. In every case, the number 3 was the last number.

3. Counting consists of matching a given set with a certain ordered subset of the natural numbers to arrive at the number of the given set.

4. 1

5. 0

6. Various correct answers are possible. One example is the set of people 22 feet tall. Such a set is called an empty set.

DEVELOPMENT OF OUR DECIMAL NUMERATION SYSTEM

Exercise Set 1, pp. 19-20

1. **a.** 4
b. 11
c. 33
d. 101
e. 200
f. 222
g. 244
h. 708
i. 555

2. 999∩∩∩∩|||||

3. **a.** 𓆼
b. 𓂭
c. Invent a new symbol. This symbol would represent 100,000.
d. Invent a new symbol.
e. 7
f. Invent a new symbol to represent one billion. (We are assuming they had already invented symbols for 1, 10, 100, 1,000, 10,000, etc.)

4. **a.** 2
b. 7
c. 27
d. 66
e. 278
f. 512
g. 1,012
h. 2,211
i. 383

5. The value of the numeral is the sum of the values of each symbol making up the numeral.

6. 23

7. 45

8. The value of the numeral is the sum of the values of each symbol making up the numeral.

Exercise Set 2, pp. 21-22

1. **a.** 12 **b.** 62 **c.** 610 **d.** 1,203 **e.** 1,281

2. **a.** 2 **b.** 12 **c.** 144 **d.** 60

Exercise Set 3, pp. 23-24

1. 7 fours + 0 ones
2. 5 fives + 3 ones
3. 2 tens + 8 ones
4. 33
5. 42

Exercise Set 4, pp. 26-27

1. a. 36
   b. 567
   c. 8,645
   d. 4,073
   e. 36
   f. 563
   g. 8,764
   h. 5,068
2. The digit 3 represents 3 x 100.
   The digit 4 represents 4 x 10.
   The digit 7 represents 7 x 1.
3. The digit 4 represents 4 x 1,000.
   The digit 9 represents 9 x 100.
   The digit 1 represents 1 x 10.
   The digit 6 represents 6 x 1.
4. The digit 3 (at left) represents 3 x 1,000.
   The digit 0 represents 0 x 100.
   The digit 3 represents 3 x 10.
   The digit 3 (at right) represents 3 x 1.

Exercise Set 5, pp. 27-28

1. a. The digit 5 represents 5 x 10,000.
   The digit 6 represents 6 x 1,000.
   The digit 3 represents 3 x 100.
   The digit 4 represents 4 x 10.
   The digit 2 represents 2 x 1.
   b. The digit 2 represents 2 x 10,000.
   The digit 0 represents 0 x 1,000.
   The digit 5 represents 5 x 100.
   The digit 1 represents 1 x 10.
   The digit 8 represents 8 x 1.

2. Multiply ten x ten x ten x ten by ten.

3. a. eight x eight; eight x eight x eight;
   eight x eight x eight x eight.
   b. seven x seven; seven x seven x seven;
   seven x seven x seven x seven.
   c. five x five; five x five x five;
   five x five x five x five.
   d. six x six; six x six x six; six x six x six x six.

Exercise Set 6, pp. 31-32

1.

PLACE-VALUE CHART

ten x ten x ten x ten 10^4	ten x ten x ten 10^3	ten x ten 10^2	base ten 10^1	ones 10^0

2. a. $100 + 40 + 6$
$1 \times 100 + 4 \times 10 + 6 \times 1$
$1 \times (10 \times 10) + 4 \times 10 + 6 \times 1$
$1 \times 10^2 + 4 \times 10^1 + 6 \times 10^0$

b. $300 + 20 + 9$
$3 \times 100 + 2 \times 10 + 9 \times 1$
$3 \times (10 \times 10) + 2 \times 10 + 9 \times 1$
$3 \times 10^2 + 2 \times 10^1 + 9 \times 10^0$

c. $7{,}000 + 100 + 40 + 6$
$7 \times 1{,}000 + 1 \times 100 + 4 \times 10 + 6$
$7 \times (10 \times 10 \times 10) + 1 \times (10 \times 10) + 4 \times 10 + 6 \times 1$
$7 \times 10^3 + 1 \times 10^2 + 4 \times 10^1 + 6 \times 10^0$

d. $30{,}000 + 3{,}000 + 400 + 10 + 2$
$3 \times 10{,}000 + 3 \times 1{,}000 + 4 \times 100 + 1 \times 10 + 2 \times 1$
$3 \times (10 \times 10 \times 10 \times 10) + 3 \times (10 \times 10 \times 10) + 4 \times (10 \times 10)$
$+ 1 \times 10 + 2 \times 1$
$3 \times 10^4 + 3 \times 10^3 + 4 \times 10^2 + 1 \times 10^1 + 2 \times 10^0$

e. $200{,}000 + 90{,}000 + 6{,}000 + 300 + 10 + 4$
$2 \times 100{,}000 + 9 \times 10{,}000 + 6 \times 1{,}000 + 3 \times 100 +$
$1 \times 10 + 4 \times 1$
$2 \times (10 \times 10 \times 10 \times 10 \times 10) + 9 \times (10 \times 10 \times 10 \times 10) +$
$6 \times (10 \times 10 \times 10) + 3 \times (10 \times 10) + 1 \times 10 + 4 \times 1$
$2 \times 10^5 + 9 \times 10^4 + 6 \times 10^3 + 3 \times 10^2 + 1 \times 10^1$
$+ 4 \times 10^0$

3.
- **a.** 436
- **b.** 6,547
- **c.** 7,658
- **d.** 5,069
- **e.** 65,436
- **f.** 4,676
- **g.** 50,060
- **h.** 4,361
- **i.** 747
- **j.** 8,567
- **k.** 963,256
- **l.** 7,369,600
- **m.** 8,003,056

4.
- **a.** 10
- **b.** 9
- **c.** 1
- **d.** 100 or 10^2
- **e.** 10
- **f.** 10^1
- **g.** 3, 3
- **h.** 4, 4
- **i.** 10, 10, 10, 10 ,10

5.
$$\begin{array}{rl} 36 = 30 + 6 & \\ \underline{27 = 20 + 7} & \\ \overline{\quad 50 + 13} & = 50 + 10 + 3 \\ & = 60 + 3 \\ & = 63 \end{array}$$

ADDITION AND ITS PROPERTIES

Exercise Set 1, p. 36

1. $A \cup B = \{a, b, c, e, i, o, u\}$.
2. $B \cup A = \{a, b, c, e, i, o, u\}$.
3. $A \cup C = \{a, b, c, f, g\}$.
4. $B \cup C = \{a, b, e, f, g, i, o, u\}$.

5. $A \cup D = \{a, b, c, u, v, w, x, y, z\}$.
6. $(A \cup B) \cup C = \{a, b, c, e, f, g, i, o, u\}$.
7. $A \cup (B \cup C) = \{a, b, c, e, f, g, i, o, u\}$.
8. $B \cup \{\ \} = \{a, e, i, o, u\} = B$.

Exercise Set 2, pp. 38-39

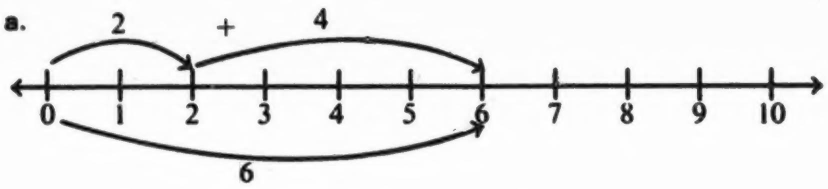
1. We choose a set A with 7 elements and a set B (disjoint from A) with 2 elements. Then $7 + 2$ is the number of elements in $A \cup B$.

2. The expressions in **b**, **c**, **e**, **f**, and **i** are meaningless. The term "union" and the symbol "∪" apply to sets, whereas "sum" and "+" are used with numbers.

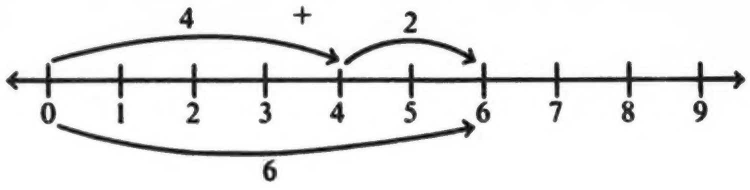
3. A and B must have exactly two elements in common.

4. No. If A and B are disjoint, $n(A) + n(B) = n(A \cup B)$. If A and B are not disjoint, $n(A) + n(B) > n(A \cup B)$.

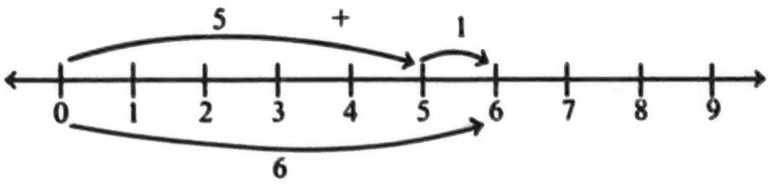
5. a.



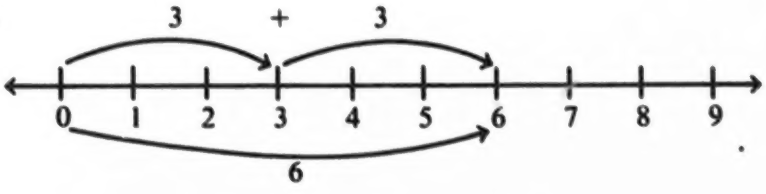
b.



c.



d.



Exercise Set 3, pp. 43-44

1. a. Addition is commutative.
b. Addition is associative.
c. Addition is commutative: 4 and 7 have been interchanged.
d. Addition is commutative: $2 + 9$ and $3 + 1$ have been interchanged.

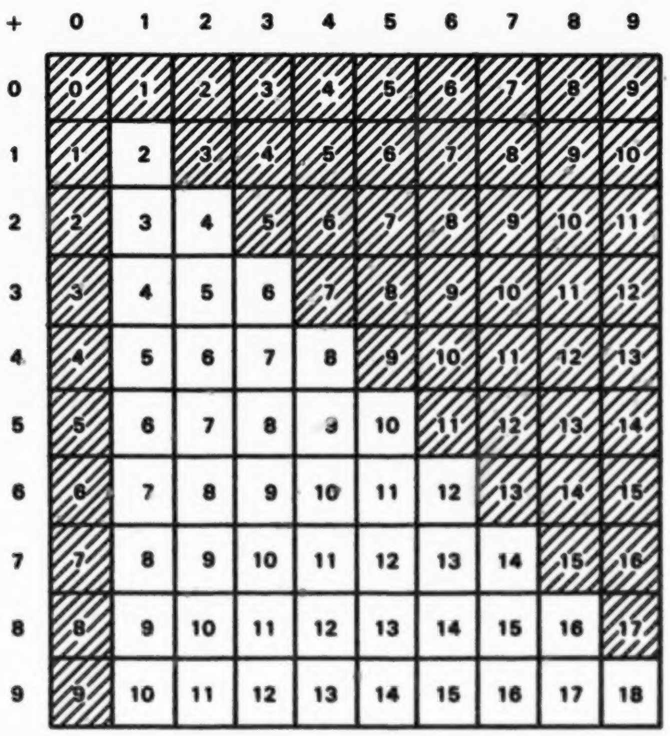
- **e.** Addition is associative.
- **f.** Addition is commutative.

2. **a.** $7 + (3 + 6) = (7 + 3) + 6 = 10 + 6.$
b. $8 + (5 + 2) = 8 + (2 + 5) = (8 + 2) + 5 = 10 + 5.$
c. $(4 + 9) + 1 = 4 + (9 + 1) = 4 + 10.$
d. $17 + (28 + 3) = 17 + (3 + 28) = (17 + 3) + 28 = 20 + 28.$
e. $(16 + 7) + (3 + 4) = (16 + 4) + (7 + 3) = 20 + 10.$

3. **a.** Yes, $a \star b = b \star a$ for all numbers *a* and *b* because 2 times the sum of *a* and *b* is the same as 2 times the sum of *b* and *a*.
b. $2 \star (3 \star 4) = 2 \star 14 = 32.$
c. No, because $(2 \star 3) \star 4 \neq 2 \star (3 \star 4).$

Exercise Set 4, p. 45

1. Consider the addition facts displayed in the form of a table. The first row of the body of the table and the first column are taken care of by the addition property of 0. The commutative property makes it unnecessary to memorize the others crossed out below. There are 45 facts left.



+	0	1	2	3	4	5	6	7	8	9
0	~~0~~	~~1~~	~~2~~	~~3~~	~~4~~	~~5~~	~~6~~	~~7~~	~~8~~	~~9~~
1	~~1~~	2	~~3~~	~~4~~	~~5~~	~~6~~	~~7~~	~~8~~	~~9~~	~~10~~
2	~~2~~	3	4	~~5~~	~~6~~	~~7~~	~~8~~	~~9~~	~~10~~	~~11~~
3	~~3~~	4	5	6	~~7~~	~~8~~	~~9~~	~~10~~	~~11~~	~~12~~
4	~~4~~	5	6	7	8	~~9~~	~~10~~	~~11~~	~~12~~	~~13~~
5	~~5~~	6	7	8	9	10	~~11~~	~~12~~	~~13~~	~~14~~
6	~~6~~	7	8	9	10	11	12	~~13~~	~~14~~	~~15~~
7	~~7~~	8	9	10	11	12	13	14	~~15~~	~~16~~
8	~~8~~	9	10	11	12	13	14	15	16	~~17~~
9	~~9~~	10	11	12	13	14	15	16	17	18

2. $a < b$ means that there is a whole number c, other than 0, such that $a + c = b$.

MULTIPLICATION AND ITS PROPERTIES

Exercise Set 1, p. 49

1. (a,e) (b,e) (c,e) (d,e)
(a,f) (b,f) (c,f) (d,f)
(a,g) (b,g) (c,g) (d,g)

2: $\{a, b\} \times \{r, s, t, u\} = \{(a, r), (a, s), (a, t), (a, u), (b, r), (b, s), (b, t), (b, u)\}$.

3. $A \times B = \{(x, r), (x, s), (x, t), (y, r), (y, s), (y, t)\}$.
$B \times A = \{(r, x), (r, y), (s, x), (s, y), (t, x), (t, y)\}$.
$A \times B$ is *not* the same set as $B \times A$. However, $A \times B$ is equivalent to $B \times A$.

4. $n(A) = 2$. $n(B) = 3$. $n(A \times B) = 6$. $n(B \times A) = 6$.

Exercise Set 2, pp. 51-52

1. 6: (red, brown) (red, black) (red, white)
(white, brown) (white, black) (white, white)

2. $2 \times 2 = 4$.

3. 4×5, or 20

Exercise Set 3, pp. 54-55

1. a. $4 \times 3 = 12$. b. $2 \times 3 = 6$. c. $6 \times 2 = 12$.

2. Various answers are possible. For example:
$A = \{a, b, c, d\}$, 4 elements.
$B = \{q, p, r, s\}$, 4 elements.
Their union is $\{a, b, c, d, q, p, r, s\}$, 8 elements.

3. a. $6 \times 3 = 18$. c. $3 \times 0 = 0$. e. $2 \times 2 = 4$.
b. $4 \times 1 = 4$. d. $6 \times 6 = 36$.

4. We should find the *product* of the two numbers 3 and 24, because a solution of this problem rests upon the use of our second approach to multiplication. Each quart of milk costs 24 cents, so the total cost for 3 quarts is $(24 + 24 + 24)$ cents, which, according to our second interpretation, is (3×24) cents. (Computation yields 72 cents as the total cost.)

Exercise Set 4, p. 57

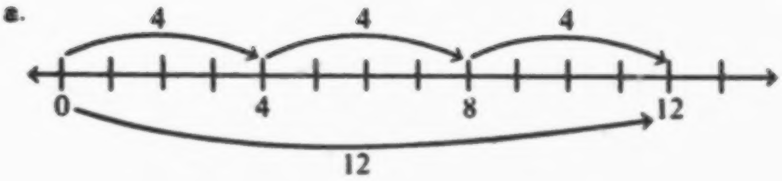
1. a. $3 \times 5 = 15$. b. $5 \times 3 = 15$. c. $5 \times 5 = 25$.

2. a.
o o
o o
o o

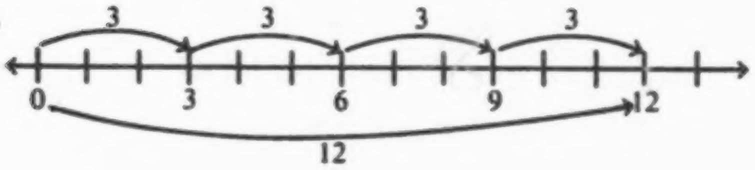
b.
x
x
x
x
x
x

c.
• • • • • • •
• • • • • • •
• • • • • • •
• • • • • • •

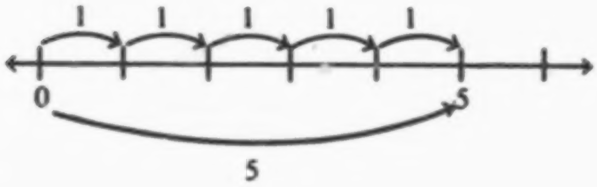
3. a.



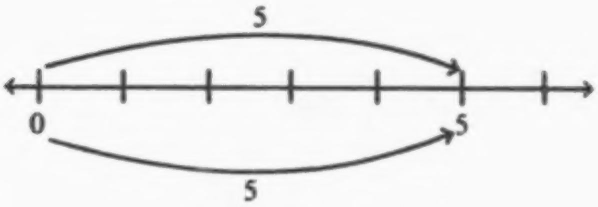
b.



c.



d.



Exercise Set 5, p. 62

1. a. $\boxed{9} \times 8 = 8 \times 9.$
   b. $4 \times \overset{\triangle}{17} = 17 \times 4.$
   c. $67 \times \boxed{87} = 87 \times 67.$
   d. $\boxed{13} \times \overset{\triangle}{17} = 17 \times 13.$
   e. $(\boxed{9} \times 6) \times 7 = (6 \times 9) \times 7.$
   f. $(16 \times 35) \times \boxed{12} = 12 \times (16 \times 35).$

2. a. $(\boxed{5} \times 6) \times 7 = 5 \times (6 \times 7).$
   b. $(8 \times \overset{\triangle}{9}) \times 9 = 8 \times (9 \times 9).$
   c. $16 \times (\boxed{8} \times 4) = (16 \times 8) \times \overset{\triangle}{4}.$

3. a. $(67 \times 50) \times 2 = 67 \times 100 = 6{,}700.$
   b. $(5 \times 13) \times 2 = (13 \times 5) \times 2 = 13 \times 10 = 130.$
   c. $8 \times 700 = (8 \times 7) \times 100 = 56 \times 100 = 5{,}600.$

4. a. $(16 - 9) - 3 = 7 - 3 = 4.$
   $16 - (9 - 3) = 16 - 6 = 10.$
   b. $18 - (7 - 4) = 18 - 3 = 15.$
   $(18 - 7) - 4 = 11 - 4 = 7.$
   c. No, subtraction is not associative.

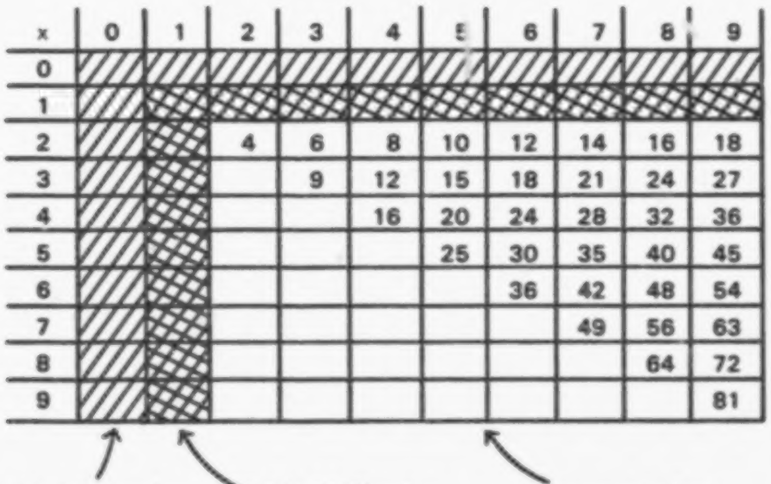
Exercise Set 6, pp. 64-65

1. a. $7 \times \boxed{0} = 0$.
   b. $0 \times 6 = \boxed{0}$.
   c. $75 \times \boxed{1} = 75$.
   d. $\boxed{10} \times 1 = 10$.
   e. $1 \times 0 = \boxed{0}$.
   f. $\boxed{0} \times 7 = \boxed{0}$.

2. a. All whole numbers
   $\boxed{0} \times 1 = \boxed{0}$, $\boxed{1} \times 1 = \boxed{1}$, $\boxed{2} \times 1 = \boxed{2}$, etc.
   b. All whole numbers
   $0 \times \boxed{0} = 0$, $0 \times \boxed{1} = 0$, $0 \times \boxed{2} = 0$, etc.
   c. 0 and 1
   $\boxed{0} \times \boxed{0} = \boxed{0}$. $\boxed{1} \times \boxed{1} = \boxed{1}$.

3. a. $3 \times 7 \times 85 \times 0 \times 96 = 0$. Multiplication property of 0.
   b. $576 \times 1 = 576$. Multiplication property of 1.
   c. $(75 - 75) \times 37 = 0$. Multiplication property of 0.

4.



x	0	1	2	3	4	5	6	7	8	9
0	▨	▨	▨	▨	▨	▨	▨	▨	▨	▨
1	▨	▨	▨	▨	▨	▨	▨	▨	▨	▨
2	▨	▨	4	6	8	10	12	14	16	18
3	▨	▨		9	12	15	18	21	24	27
4	▨	▨			16	20	24	28	32	36
5	▨	▨				25	30	35	40	45
6	▨	▨					36	42	48	54
7	▨	▨						49	56	63
8	▨	▨							64	72
9	▨	▨								81

Multiplication Property of 0
Multiplication Property of 1
Commutative Property

SUBTRACTION

Exercise Set 1, p. 70

1. a. Yes, B is a subset of A since $B = \{a, e, i, o, u\}$ and the letters a, e, i, o, and u are elements of $A = \{a, e, f, i, j, o, p, u\}$.
   The subset of A composed of elements not in B is $\{f, j, p\}$.
   b. Yes, B is a subset of A.
   The remaining subset of A is {red, white, blue}.
   c. Yes, B is a subset of A.
   The remaining subset of A is {California, Oregon, Washington}.

d. Yes, B is a subset of A even though $B = A$.
The remaining subset is the empty set, { }.

e. Yes, B is a subset of A; B is the empty set, and the empty set is a subset of every set.
The remaining subset is {△, □, ○}, A itself.

2. **a.** $8 - 5 = 3$.
b. $4 - 1 = 3$.
c. $5 - 2 = 3$.
d. $3 - 3 = 0$.
e. $3 - 0 = 3$.

3. $B \cup C = A$ since B and C together consist of all the elements of A. Let A = {Jim, Jack, Jerry, Jane} and B = {Jane}. Then the subset C of A composed of elements not in B is {Jack, Jerry, Jim} and

$B \cup C$ = {Jane} ∪ {Jack, Jerry, Jim}
= {Jane, Jack, Jerry, Jim} = A.

Exercise Set 2, p. 73

1. **a.** $6 = 10 - 4$. $4 = 10 - 6$.
b. $1 = 9 - 8$. $8 = 9 - 1$.
c. $0 = 12 - 12$. $12 = 12 - 0$.
d. $4 = 18 - 14$. $14 = 18 - 4$.
e. $154 = 221 - 67$. $67 = 221 - 154$.

2. **a.** $12 = 5 + 7$ (or $12 = 7 + 5$).
b. $6 = 0 + 6$ (or $6 = 6 + 0$).
c. $8 = 0 + 8$ (or $8 = 8 + 0$).
d. $10 = 5 + 5$.
e. $74 = 7 + 67$ (or $74 = 67 + 7$).

3. **a.** $\boxed{9} = 12 - 3$.
b. $\boxed{1} = 7 - 6$.
c. $\boxed{0} = 12 - 12$.
d. $\boxed{1} = 15 - 14$.
e. $\boxed{67} = 95 - 28$.

4. **a.** $9 + \boxed{7} = 16$ (or $\boxed{7} + 9 = 16$).
b. $1 + \boxed{3} = 4$ (or $\boxed{3} + 1 = 4$).
c. $2 + \boxed{4} = 6$ (or $\boxed{4} + 2 = 6$).
d. $\boxed{0} + 9 = 9$ (or $9 + \boxed{0} = 9$).
e. $\boxed{3} + 72 = 75$ (or $72 + \boxed{3} = 75$).

Exercise Set 3, p. 75

1. **a.** $3 - 2 \ \textcircled{\ne}\ 2 - 3$.
b. $(6 - 4) - 0 \ \textcircled{=}\ 6 - (4 - 0)$.
c. $8 \div 8 \ \textcircled{=}\ 8 \div 8$.
d. $86 \times 74 \ \textcircled{=}\ 74 \times 86$.
e. $6 - (4 - 1) \ \textcircled{\ne}\ (6 - 4) - 1$.

2. **a.** $8 - (4 - 1) = 5$.
b. $(24 \div 6) \div 2 = 2$.
c. $(12 - 7) - 0 = 5$ and $12 - (7 - 0) = 5$.
d. $(9 \times 4) \times 2 = 72$ and $9 \times (4 \times 2) = 72$.
e. $2 \times (4 + 7) = 22$.

DIVISION

Exercise Set 1, p. 82

1\. a. △ △ △ △ △ △
△ △ △ △ △ △

c. ★ ★
★ ★
★ ★
★ ★
★ ★

b. ☆ ☆ ☆ ☆
☆ ☆ ☆ ☆
☆ ☆ ☆ ☆
☆ ☆ ☆ ☆

d. ○ ○ ○ ○ ○ ○

2\. a. 6 columns
b. 4 columns
c. 2 columns
d. 6 columns

3\. a. $15 \div 5 = 3.$
$15 \div 3 = 5.$
b. $24 \div 6 = 4.$
$24 \div 4 = 6.$
c. $16 \div 2 = 8.$
$16 \div 8 = 2.$
d. $36 \div 9 = 4.$
$36 \div 4 = 9.$

Exercise Set 2, pp. 84-86

1\. a. $6 = 42 \div 7.$ $7 = 42 \div 6.$
b. $3 = 3 \div 1.$ $1 = 3 \div 3.$
c. $10 = 90 \div 9.$ $9 = 90 \div 10.$
d. $13 = 130 \div 10.$ $10 = 130 \div 13.$
e. $4 = 32 \div 8.$ $8 = 32 \div 4.$

2\. a. $6 \times \boxed{5} = 30.$
b. No whole-number factor. (A rational number will fit.)
c. $\boxed{11} \times 9 = 99.$
d. No whole-number factor. (A rational number will fit.)
e. $2 \times \boxed{1} = 2.$
f. $3 \times \boxed{0} = 0.$
g. No whole-number factor. (A rational number will fit.)
h. No whole-number factor. (A rational number will fit.)
i. $\boxed{0} \times 15 = 0.$
j. $4 \times \boxed{38} = 152.$

3\. a. $8 \div 2 = \boxed{4}.$ $8 = \boxed{4} \times 2$ or $8 = 2 \times \boxed{4}.$
b. $6 \div 6 = \boxed{1}.$ $6 = \boxed{1} \times 6$ or $6 = 6 \times \boxed{1}.$
c. $12 \div 1 = \boxed{12}.$ $\boxed{12} = 1 \times 12$ or $12 = \boxed{12} \times 1.$
d. $0 \div 8 = \boxed{0}.$ $0 = 8 \times \boxed{0}$ or $0 = \boxed{0} \times 8.$
e. $55 \div 11 = \boxed{5}.$ $55 = 11 \times \boxed{5}$ or $55 = \boxed{5} \times 11.$

4\. Any whole number will complete $0 \times \square = 0.$
$0 \times 0 = 0;\ 0 \times 1 = 0;\ 0 \times 2 = 0;\ 0 \times 3 = 0;$ etc.

5. a. $62 = (7 \times \boxed{8}) + \triangle\!\!\!\!{\scriptstyle 6}$.
   b. $6 = (4 \times \boxed{1}) + \triangle\!\!\!\!{\scriptstyle 2}$.
   c. $5 = (7 \times \boxed{0}) + \triangle\!\!\!\!{\scriptstyle 5}$.
   d. $55 = (11 \times \boxed{5}) + \triangle\!\!\!\!{\scriptstyle 0}$.
   e. $57 = (7 \times \boxed{8}) + \triangle\!\!\!\!{\scriptstyle 1}$.

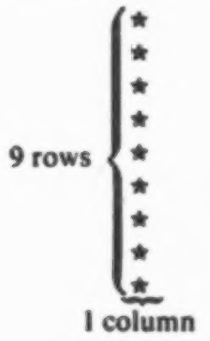
Exercise Set 3, p. 87

1. a. $6 \div 2 \bigcirc\!\!\!\!\!\!\neq\; 2 \div 6$.
   b. $(6 \div 2) \div 1 \bigcirc\!\!\!\!\!\!=\; 6 \div (2 \div 1)$.
   c. $(16 \div 4) \div 2 \bigcirc\!\!\!\!\!\!\neq\; 16 \div (4 \div 2)$.
   d. $(12 \div 6) \times 2 \bigcirc\!\!\!\!\!\!\neq\; 12 \div (6 \times 2)$.
   e. $12 \times (6 \div 2) \bigcirc\!\!\!\!\!\!=\; (12 \times 6) \div 2$.

2. a. $8 \div (4 \times 2) = 1$.
   b. $8 \div (4 \div 2) = 4$.
   c. $12 \div (3 + 1) = 3$.
   d. $(12 \div 3) - 1 = 3$.
   e. $12 \div (3 \times 2) = 2$.

Exercise Set 4, pp. 90-91

1. An array of 9 rows and 9 elements has 1 column:



$9 \div 9 = 1$.

2. The expressions in **a**, **b**, **d**, and **f**, because
   a. $0 \div 7 = 0$.
   b. $14 - 14 = 0$.
   c. $9 - 0 = 9$.
   d. $1 \times 0 = 0$.
   e. $0 \div 0$ is meaningless.
   f. $0 \div 1 = 0$.

3. The expression in **e** because $0 \div a = 0$, provided $a \neq 0$.

ADDITION AND SUBTRACTION ALGORITHMS

Exercise Set 1, p. 99

1. **a.**

$$\begin{aligned}53 + 15 &= (50 + 3) + (10 + \boxed{5})\\ &= (50 + 10) + (3 + 5)\\ &= 60 + 8\\ &= 68.\end{aligned}$$

b.

$$\begin{aligned}27 + 35 &= (\boxed{20} + 7) + (30 + 5)\\ &= (\boxed{20} + 30) + (7 + 5)\\ &= 50 + 12\\ &= 50 + (10 + 2)\\ &= (50 + 10) + 2\\ &= 60 + 2\\ &= 62.\end{aligned}$$

2. **a.**

$$\begin{aligned}19 + 67 &= (10 + 9) + (60 + 7)\\ &= (10 + 60) + (9 + 7)\\ &= 70 + 16\\ &= 70 + (10 + 6)\\ &= (70 + 10) + 6\\ &= 80 + 6\\ &= 86.\end{aligned}$$

b.

$$\begin{aligned}173 + 8 &= (100 + 70 + 3) + 8\\ &= 100 + 70 + (3 + 8)\\ &= 100 + 70 + 11\\ &= 100 + 70 + (10 + 1)\\ &= 100 + (70 + 10) + 1\\ &= 100 + 80 + 1\\ &= 181.\end{aligned}$$

c.

$$\begin{aligned}231 + 36 &= (200 + 30 + 1) + (30 + 6)\\ &= 200 + (30 + 30) + (1 + 6)\\ &= 200 + 60 + 7\\ &= 267.\end{aligned}$$

d.

$$\begin{aligned}97 + 24 &= (90 + 7) + (20 + 4)\\ &= (90 + 20) + (7 + 4)\\ &= 110 + 11\\ &= (100 + 10) + (10 + 1)\\ &= 100 + (10 + 10) + 1\\ &= 100 + 20 + 1\\ &= 121.\end{aligned}$$

e.

$$\begin{aligned}208 + 523 &= (200 + 8) + (500 + 20 + 3)\\ &= (200 + 500) + 20 + (8 + 3)\\ &= 700 + 20 + 11\end{aligned}$$

$$\begin{aligned}
&= 700 + 20 + (10 + 1)\\
&= 700 + (20 + 10) + 1\\
&= 700 + 30 + 1\\
&= 731.
\end{aligned}$$

f. $$\begin{aligned}
145 + 278 &= (100 + 40 + 5) + (200 + 70 + 8)\\
&= (100 + 200) + (40 + 70) + (5 + 8)\\
&= 300 + 110 + 13\\
&= 300 + (100 + 10) + (10 + 3)\\
&= 400 + 20 + 3\\
&= 423.
\end{aligned}$$

Exercise Set 2, p. 102

1. $$\begin{array}{rcl}
78 &=& 70 + 8\\
23 &=& 20 + 3\\
\hline
&& 50 + 5 = 55.
\end{array}$$

2. $$\begin{array}{rcrcrcl}
63 &=& 60 + 3 &=& 50 &+& 13\\
7 &=& 7 &=& && 7\\
\hline
&&&& 50 &+& 6 = 56.
\end{array}$$

3. $$\begin{array}{rcrcrcl}
52 &=& 50 + 2 &=& 40 &+& 12\\
39 &=& 30 + 9 &=& 30 &+& 9\\
\hline
&&&& 10 &+& 3 = 13.
\end{array}$$

4. $$\begin{array}{rcrcrcl}
348 &=& 300 + 40 + 8 &=& 200 &+& 140 + 8\\
92 &=& 90 + 2 &=& && 90 + 2\\
\hline
&&&& 200 &+& 50 + 6 = 256.
\end{array}$$

5. $$\begin{array}{rcrcrcrcrcrcl}
403 &=& 400 \qquad + 3 &=& 300 + 100 + 3 &=& 300 &+& 90 &+& 13\\
126 &=& 100 + 20 + 6 &=& 100 + 20 + 6 &=& 100 &+& 20 &+& 6\\
\hline
&&&&&& 200 &+& 70 &+& 7 = 277.
\end{array}$$

6. $$\begin{array}{rcrcrcrcl}
500 &=& 400 + 100 \qquad &=& 400 &+& 90 &+& 10\\
278 &=& 200 + 70 + 8 &=& 200 &+& 70 &+& 8\\
\hline
&&&& 200 &+& 20 &+& 2 = 222.
\end{array}$$

Exercise Set 3, p. 103

1. **a.** Add 3 to both 629 and 297.

$$\begin{aligned}
629 - 297 &= (629 + 3) - (297 + 3)\\
&= 632 - 300 = 332.
\end{aligned}$$

b. Add 5 to both 4,384 and 1,995.

$$\begin{aligned}
4{,}384 - 1{,}995 &= (4{,}384 + 5) - (1{,}995 + 5)\\
&= 4{,}389 - 2{,}000 = 2{,}389.
\end{aligned}$$

2. This amounts to adding 10 to both 53 and 26. In one case we add it to the ones and in the other to the tens:

$$\begin{array}{lclcl}
53 - 26 &=& (53 + 10) - (26 + 10) &&\\
53 + 10 &=& 50 + (10 + 3) &=& 50 + 13\\
26 + 10 &=& (20 + 10) + 6 &=& 30 + 6\\
\hline
&&&& 20 + 7.
\end{array}$$

Exercise Set 4, p. 108

1. a. $\begin{array}{r} 32_{\text{five}} \\ \underline{43_{\text{five}}} \\ 130_{\text{five}} \end{array}$ b. $\begin{array}{r} 232_{\text{five}} \\ \underline{14_{\text{five}}} \\ 301_{\text{five}} \end{array}$ c. $\begin{array}{r} 132_{\text{five}} \\ \underline{204_{\text{five}}} \\ 341_{\text{five}} \end{array}$ d. $\begin{array}{r} 1304_{\text{five}} \\ \underline{243_{\text{five}}} \\ 2102_{\text{five}} \end{array}$

2.

BASE-EIGHT ADDITION TABLE

(All Entries To Be Interpreted as Base-Eight Numerals)

+	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	2	3	4	5	6	7	10
2	2	3	4	5	6	7	10	11
3	3	4	5	6	7	10	11	12
4	4	5	6	7	10	11	12	13
5	5	6	7	10	11	12	13	14
6	6	7	10	11	12	13	14	15
7	7	10	11	12	13	14	15	16

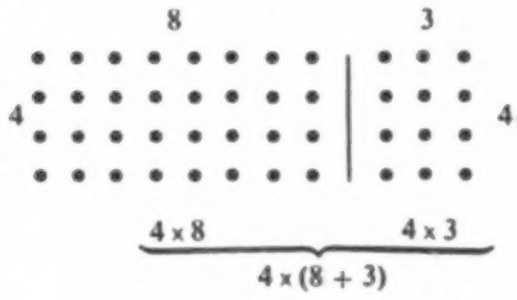
a. $\begin{array}{r} 42_{\text{eight}} \\ \underline{15_{\text{eight}}} \\ 57_{\text{eight}} \end{array}$ b. $\begin{array}{r} 53_{\text{eight}} \\ \underline{17_{\text{eight}}} \\ 72_{\text{eight}} \end{array}$ c. $\begin{array}{r} 63_{\text{eight}} \\ \underline{120_{\text{eight}}} \\ 203_{\text{eight}} \end{array}$ d. $\begin{array}{r} 234_{\text{eight}} \\ \underline{355_{\text{eight}}} \\ 611_{\text{eight}} \end{array}$

MULTIPLICATION ALGORITHMS AND THE DISTRIBUTIVE PROPERTY

Exercise Set 1, pp. 114-15

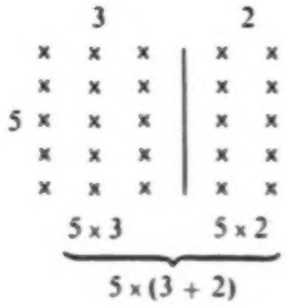
1. a. $8 \times (6 + 3) = (8 \times 6) + (8 \times 3)$.
   b. $6 \times (4 + 7) = (6 \times 4) + (6 \times 7)$.
   c. $3 \times (6 + 5) = (3 \times 6) + (3 \times 5)$.
   d. $6 \times (7 + 8) = (6 \times 7) + (6 \times 8)$.

2. a.



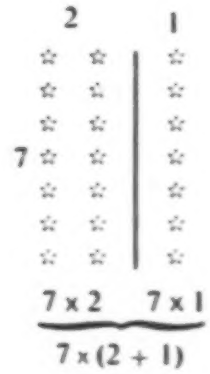
$4 \times (8 + 3) = (4 \times 8) + (4 \times 3)$.

b.


3
2
5
5 × 3
5 × 2
5 × (3 + 2)

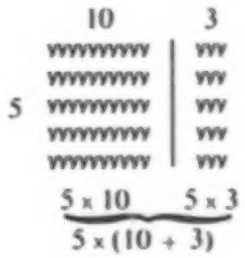
$5 \times (3 + 2) = (5 \times 3) + (5 \times 2)$.

c.


2
1
7
7 × 2
7 × 1
7 × (2 + 1)

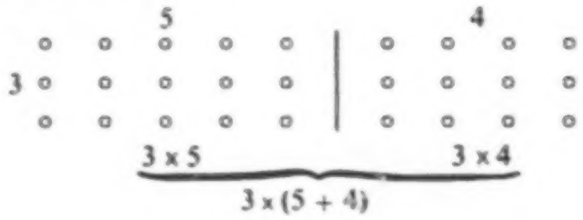
$7 \times (2 + 1) = (7 \times 2) + (7 \times 1)$.

d.


10
3
5
5 × 10
5 × 3
5 × (10 + 3)

$5 \times (10 + 3) = (5 \times 10) + (5 \times 3)$.

3. $3 \times (5 + 4) = (3 \times 5) + (3 \times 4)$.


5
4
3
3 × 5
3 × 4
3 × (5 + 4)

4. **a.** $5 + 18 = 23$.
b. $5 + (2 \times 9) = 23$.
c. $(5 + 2) \times (5 + 9) = 7 \times 14 = 98$.

No, addition is not distributive over multiplication. If it were, $5 + (2 \times 9)$ would equal $(5 + 2) \times (5 + 9)$.

Exercise Set 2, pp. 119-20

1. a. $3 \times (4 + 5) = (3 \times 4) + (3 \times 5)$. $\underline{\ \text{D}\ }$
   The 3 is "distributed" over the 4 + 5.

   b. $3 \times (4 + 5) = (4 + 5) \times 3$. $\underline{\ \text{C}\ }$
   The 3 and the 4 + 5 are interchanged, or "commuted."

   c. $(4 + 5) \times 3 = (4 \times 3) + (5 \times 3)$. $\underline{\ \text{D}\ }$
   This is the second form of the distributive property, where the "distributivity" is done "from the right."

   d. $(4 \times 3) + (5 \times 3) = (5 \times 3) + (4 \times 3)$. $\underline{\ \text{C}\ }$
   The 4 x 3 and the 5 x 3 are commuted.

   e. $3 \times (4 \times 5) = (3 \times 4) \times 5$. $\underline{\ \text{A}\ }$
   This is an instance of the associative property of multiplication.

2. b.
$$\begin{aligned} 7 \times 15 &= 7 \times (6 + 9) \\ &= (7 \times 6) + (7 \times 9) \\ &= 42 + 63 \\ &= 105, \end{aligned}$$
   or
$$\begin{aligned} 7 \times 15 &= 7 \times (7 + 8) \\ &= (7 \times 7) + (7 \times 8) \\ &= 49 + 56 \\ &= 105, \end{aligned}$$
   etc.

   c.
$$\begin{aligned} 13 \times 8 &= (6 + 7) \times 8 \\ &= (6 \times 8) + (7 \times 8) \\ &= 48 + 56 \\ &= 104, \end{aligned}$$
   or
$$\begin{aligned} 13 \times 8 &= (9 + 4) \times 8 \\ &= (9 \times 8) + (4 \times 8) \\ &= 72 + 32 \\ &= 104, \end{aligned}$$
   etc.

   d.
$$\begin{aligned} 26 \times 6 &= (9 + 9 + 8) \times 6 \\ &= (9 \times 6) + (9 \times 6) + (8 \times 6) \\ &= 54 + 54 + 48 \\ &= 156. \end{aligned}$$

3. b.
$$\begin{aligned} (98 \times 2) + (2 \times 2) &= (98 + 2) \times 2 \\ &= 100 \times 2 \\ &= 200. \end{aligned}$$

   c.
$$\begin{aligned} (6 \times 189) + (6 \times 11) &= 6 \times (189 + 11) \\ &= 6 \times 200 \\ &= 1{,}200. \end{aligned}$$

Exercise Set 3, pp. 123-24

1. **b.** $259 = 200 + 50 + 9.$

c. $35 = 30 + 5.$

d. $4{,}560 = 4{,}000 + 500 + 60 + 0$ or $4{,}000 + 500 + 60.$

e. $408 = 400 + 8$ or $400 + 0 + 8.$

2. **a.**

$$\begin{aligned} 7 \times 16 &= 7 \times (10 + 6) \\ &= (7 \times 10) + (7 \times 6) \\ &= 70 + 42 \\ &= 112. \end{aligned}$$

b.

$$\begin{aligned} 5 \times 97 &= 5 \times (90 + 7) \\ &= (5 \times 90) + (5 \times 7) \\ &= 450 + 35 \\ &= 485. \end{aligned}$$

c.

$$\begin{aligned} 3 \times 655 &= 3 \times (600 + 50 + 5) \\ &= (3 \times 600) + (3 \times 50) + (3 \times 5) \\ &= 1{,}800 + 150 + 15 \\ &= 1{,}965. \end{aligned}$$

3. **a.**

$$\begin{array}{rl} 147 & \\ \times\ 4 & \\ \hline 28 & \leftarrow 4 \times 7 \\ 160 & \leftarrow 4 \times 40 \\ 400 & \leftarrow 4 \times 100 \\ \hline 588 & \leftarrow 4 \times 147 \end{array}$$

b.

$$\begin{array}{rl} 1{,}508 & \\ \times\ 3 & \\ \hline 24 & \leftarrow 3 \times 8 \\ 1{,}500 & \leftarrow 3 \times 500 \\ 3{,}000 & \leftarrow 3 \times 1{,}000 \\ \hline 4{,}524 & \leftarrow 3 \times 1{,}508 \end{array}$$

Exercise Set 4, pp. 126-27

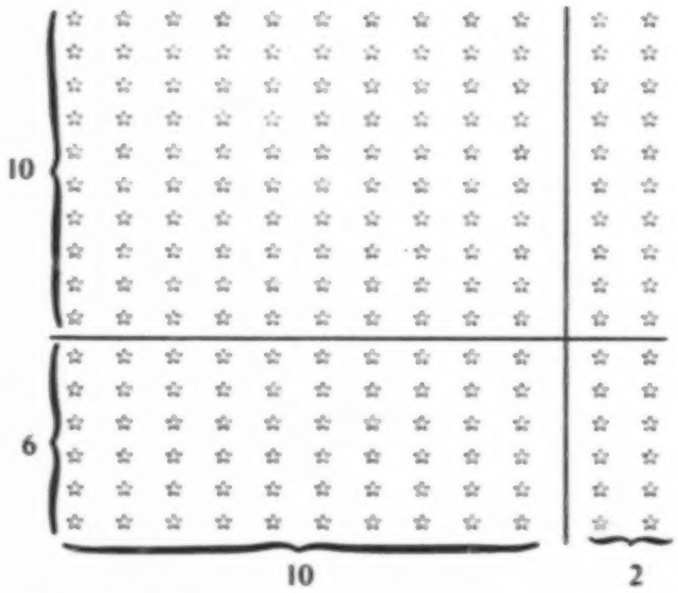
1. **a.**

$$\begin{aligned} 21 \times 32 &= (20 + 1) \times 32 \\ &= (20 \times 32) + (1 \times 32) \\ &= (10 \times 2 \times 32) + (1 \times 32) \\ &= (10 \times 64) + 32 \\ &= 640 + 32 \\ &= 672. \end{aligned}$$

b.

$$\begin{aligned} 34 \times 156 &= (30 + 4) \times 156 \\ &= (30 \times 156) + (4 \times 156) \\ &= (10 \times 3 \times 156) + (4 \times 156) \\ &= (10 \times 468) + 624 \\ &= 4{,}680 + 624 \\ &= 5{,}304. \end{aligned}$$

2.



DIVISION ALGORITHMS

Exercise Set 1, pp. 135-36

1. a. $23 = (\boxed{4} \times 5) + \triangle 3$.
   b. $5 = (\boxed{0} \times 7) + \triangle 5$.
   c. $6 = (\boxed{1} \times 6) + \triangle 0$.
   d. $97 = (\boxed{10} \times 9) + \triangle 7$.
   e. $38 = (\boxed{6} \times 6) + \triangle 2$.
   f. $31 = (\boxed{7} \times 4) + \triangle 3$.
   g. $0 = (\boxed{0} \times 8) + \triangle 0$.
   h. $61 = (\boxed{3} \times 20) + \triangle 1$.

2. The number for the triangle is less than the second number represented on the right-hand side of the equal sign.

3. No.

Exercise Set 2, p. 137

1. $q = 12, r = 3$.
2. $q = 10, r = 0$.
3. $q = 1, r = 4$.
4. $q = 0, r = 6$.
5. $q = 1, r = 0$.

Exercise Set 3, pp. 140-41

1. All solutions are correct. A total of 86 nines may be subtracted from 781. The order in which the nines are subtracted does not affect the answer.

2. a. $q = 52, r = 9$.
   b. $q = 27, r = 2$.
   c. $q = 104, r = 0$.
   d. $q = 34, r = 73$.

Exercise Set 4, p. 142

1. $q = 15, r = 7.$
2. $q = 34, r = 18.$
3. $q = 62, r = 30.$
4. $q = 105, r = 0.$

Exercise Set 5, p. 147

1. $q = 585, r = 3.$
2. $q = 173, r = 22.$
3. $q = 192, r = 16.$
4. $q = 400, r = 21.$

THE WHOLE-NUMBER SYSTEM—KEY IDEAS

Exercise Set 1, pp. 152-53

1. The following pairs of sets are equivalent: A and B, A and E, B and E; C and G; D and F.

NOTE.—Any set is, of course, also equivalent to itself; for example, A is equivalent to A, B is equivalent to B, etc.

2. Two sets are equivalent if they can be matched with a one-to-one correspondence.

3. $n(A) = 1.$ $n(C) = 2.$ $n(E) = 1.$ $n(G) = 2.$
$n(B) = 1.$ $n(D) = 3.$ $n(F) = 3.$ $n(H) = 0.$

4. Two sets are assigned the same number when they are equivalent.

5. A and F, B and D, D and G.

6. Two sets are said to be disjoint if there is no element common to both sets.

7. Find a set having the number property 2, such as $A = \{r, s\}$, and a set disjoint from A with the number property 3, such as $B = \{u, v, w\}$. Then count the number of elements in $A \cup B$.

$$\begin{aligned} 2 + 3 &= n(A \cup B) \\ &= n(\{r, s, u, v, w\}) \\ &= 5. \end{aligned}$$

8. Let A be a set such that $n(A) = a$. Let B be a set, disjoint from A, with $n(B) = $ b. Then

$$a + b = n(A \cup B).$$

9. **a.** $A \times B = \{(a, 2)\}.$
- **b.** $A \times C = \{(a, \text{John}), (a, \text{Mary})\}.$
- **c.** $C \times D = \{(\text{John}, 0), (\text{John}, 1), (\text{John}, 2), (\text{Mary}, 0), (\text{Mary}, 1), (\text{Mary}, 2)\}.$
- **d.** $D \times C = \{(0, \text{John}), (1, \text{John}), (2, \text{John}), (0, \text{Mary}), (1, \text{Mary}), (2, \text{Mary})\}.$
- **e.** $D \times E = \{(0, 7), (1, 7), (2, 7)\}.$
- **f.** $D \times F = \{(0, a), (0, b), (0, c), (1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}.$
- **g.** $D \times G = \{(0, 1), (0, \text{blue}), (1, 1), (1, \text{blue}), (2, 1), (2, \text{blue})\}.$
- **h.** $D \times H = \{\ \}.$

10. $n(A \times B) = 1,\quad 1 \times 1 = 1.\qquad n(D \times E) = 3,\quad 3 \times 1 = 3.$
$n(A \times C) = 2,\quad 1 \times 2 = 2.\qquad n(D \times F) = 9,\quad 3 \times 3 = 9.$
$n(C \times D) = 6,\quad 2 \times 3 = 6.\qquad n(D \times G) = 6,\quad 3 \times 2 = 6.$
$n(D \times C) = 6,\quad 3 \times 2 = 6\qquad n(D \times H) = 0,\quad 3 \times 0 = 0.$

11. Let sets A and B be chosen such that $n(A) = a$, $n(B) = b$. Then

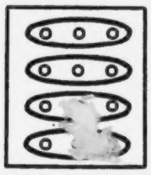
$$a \times b = n(A \times B).$$

12. **a.** $2 + 3 = 5$ or $3 + 2 = 5$.
b. $5 - 2 = 3$ and $5 \quad 3 = 2$.

13. Let A be a set such that $n(A) = a$. Let B be a subset of A such that $n(B) = b$. Let C be those elements of A that are not in B. Then

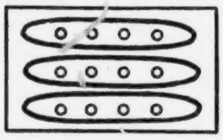
$$a - b = n(C).$$

14.



$12 \div 3 = 4.$

There are 4 sets, each having 3 elements.



$12 \div 3 = 4.$

When a set of 12 elements is partitioned into 3 disjoint equivalent sets, each set has 4 elements.

15. **a.** $\{4\}$
b. $\{10\}$
c. $\{\ \}$
d. $\{2\}$
e. $\{0, 1, 2, 3\}$
f. $\{3, 4, 5, 6, 7, 8, 9\}$
g. $\{0, 1, 2\}$
h. $\{0, 3, 6, 9, 12, 15, 18\}$
i. $\{0, 6, 12, 18\}$

Exercise Set 2, pp. 157-61

(*There are correct solutions other than those given here.*)

1. **a.** Addition is commutative.
b. Addition is associative.
c. Addition is associative:

$$\begin{aligned} 35 + 2 &= (30 + 5) + 2 \\ &= 30 + (5 + 2) \\ &= 30 + 7 \\ &= 37. \end{aligned}$$

d. Addition is associative:

$$\begin{aligned} 35 + 9 &= (30 + 5) + 9 \\ &= 30 + (5 + 9) \\ &= 30 + (14) \\ &= 30 + (10 + 4) \\ &= (30 + 10) + 4 \\ &= 40 + 4 \\ &= 44. \end{aligned}$$

e. *Solution I:*
$(a + b) - c = a + (b - c)$, for b not less than c. Thus

$$\begin{aligned}(30 + 5) - 2 &= 30 + (5 - 2)\\ &= 30 + 3\\ &= 33.\end{aligned}$$

Solution II:

$(a + b) - b = a$, from our definition of difference. Thus

$$\begin{aligned}35 - 2 &= (33 + 2) - 2\\ &= 33.\end{aligned}$$

f. $(a + b) - c = a + (b - c)$ provided b is not less than c:

$$\begin{aligned}35 - 9 &= (20 + 15) - 9\\ &= 20 + (15 - 9)\\ &= 20 + 6\\ &= 26.\end{aligned}$$

g. $a - b = (a + c) - (b + c)$ For proof, see Exercise 7 below.

h. Multiplication distributes over addition.

$$\begin{aligned}3 \times (32) &= 3 \times (30 + 2)\\ &= (3 \times 30) + (3 \times 2)\\ &= 90 + 6\\ &= 96.\end{aligned}$$

i. Multiplication distributes over addition, and addition is associative.

$$\begin{aligned}3 \times (34) &= 3 \times (30 + 4)\\ &= (3 \times 30) + (3 \times 4)\\ &= 90 + 12\\ &= 90 + (10 + 2)\\ &= (90 + 10) + 2\\ &= 100 + 2\\ &= 102.\end{aligned}$$

j. Multiplication is commutative.

k. Multiplication is associative.

l. For every number a, $a \times 0 = 0$ from the multiplication property of 0.

m. Distributive property.

n. $12 \div 3$ is by definition that (unique) number which satisfies the sentence $\square \times 3 = 12$. Since 4 is the only number which fits this sentence, it follows that $12 \div 3 = 4$.

o. $12 \div 1$ is by definition that (unique) number which satisfies the sentence $\square \times 1 = 12$. Since 12 is the only number which fits this sentence, it follows that $12 \div 1 = 12$.

p. Expressions of the form "$a \div 0$" where a is a whole number are meaningless, as they never name a specific whole number.

2. a. $678 = \boxed{6} \times 100 + \overset{7}{\triangle} \times 10 + \overset{8}{\triangledown} \times 1.$

b. $\boxed{20} + 38 = 78 - \boxed{20}.$

3. *Solution I:* If A and B are disjoint sets having a elements and b elements respectively, then $a + b$ is the number of elements in $A \cup B$. Now $(a + b) - b$ is, by definition, the number of elements of $A \cup B$ that are *not* in B. But, if we remove from $A \cup B$ the elements of B, we are left with set A. Hence

$$(a + b) - b = a.$$

Solution II: Consider the sentence

$$\square + b = a + b.$$

The missing addend for the frame is a, because $a + b = a + b$. But, from our definition of a difference, this missing addend is also $(a + b) - b$. Since there can be just one missing addend for the first sentence, it follows that

$$(a + b) - b = a.$$

Solution III: Suppose a number S is such that

$$S - b = a.$$

By definition of a difference, $S - b$ is the missing addend in the sentence

$$\square + b = S.$$

But we already know that this missing addend $S - b$ is actually a. Consequently

$$a + b = S,$$

showing that the number S is $a + b$. Using this value for S in the sentence

$$S - b = a,$$

we obtain

$$(a + b) - b = a.$$

This generalization is conveniently expressed as follows:

A sum $(a + b)$ less one of its addends (b) is the other addend (a).

4. By definition, $a - b$ is the missing addend in the sentence $\square + b = a$. Consequently, $(a - b) + b = a$.

5. $(a + b) - c$ $= (a + [(b - c) + c]) - c$ [since $b = (b - c) + c$, by Exercise 4]
$= ([a + (b - c)] + c) - c$ (since addition is associative)
$= a + (b - c)$, by Exercise 3.

6. $(a - b) - c$ $= [([a - \overline{(b + c)}] + \overline{(b + c)}) - b] - c$ (see Exercise 4)
$= ([([a - (b + c)] + c) + b] - b) - c$ (addition is commutative and associative)
$= ([a - (b + c)] + c) - c$ (see Exercise 3)
$= a - (b + c)$ (see Exercise 3).

7. $(a + c) - (b + c)$ $= ([(a - b) + b] + c) - (b + c)$ (see Exercise 4)
$= [(a - b) + (b + c)] - (b + c)$ (addition is associative)
$= a - b$ (see Exercise 3).

8. $(a - c) - (b - c)$ $= [(a - c) + c] - [(b - c) + c]$ (see Exercise 7)
$= a - b$ (see Exercise 4).

9. $(a + b) - (c + d) = (\overline{[a - c] + c} + \overline{[b - d] + d}) - (c + d)$ (see Exercise 4)
$= (\overline{[a - c] + [b - d]} + [c + d]) - (c + d)$ (rearrangement)
$= (a - c) + (b - d)$ (see Exercise 3).

10. $a - (c - d)$ $= [(a - c) + c] - (c - d)$ (see Exercise 4)
$= [(a - c) + (c - d) + d] - (c - d)$ (see Exercise 4)
$= [(a - c) + d + (c - d)] - (c - d)$ (rearrangement)
$= (a - c) + d.$

11. $(a-b)+c = c+(a-b)$ (addition is commutative)
$= (c+a)-b$ (see Exercise 5)
$= (a+c)-b$ (addition is commutative).

12. $a+b = [(a+c)-c]+b$ (see Exercise 4)
$= [(a+c)+b]-c$ (see Exercise 11)
$= (a+c)+(b-c)$ (see Exercise 5).

13. (1) $a \times b = a \times b.$	For specific values of a and b, $a \times b$ names but one whole number.
$(a \times b) \div b = a.$	A product $a \times b$ divided by one of its factors is the other factor.
(2) $a \div b = a \div b.$	For specific values of a and b, $a \div b$ names but one whole number ($b \neq 0$).
$(a \div b) \times b = a.$	In the expression $a \div b = a \div b$ consider a to be a product and b one of its factors.

14. Corresponding generalizations will be denoted by the symbol "′" next to the numeral. For example, the generalization corresponding to 4 will be denoted by 4′.

4′: For any whole numbers a and b with $a \div b$ a whole number,
$$(a \div b) \times b = a.$$

5′: For any whole numbers a, b, and c with $b \div c$ a whole number,
$$(a \times b) \div c = a \times (b \div c).$$

6′: For any whole numbers a, b, and c with $a \div (b \times c)$ a whole number,
$$(a \div b) \div c = a \div (b \times c).$$

7′: For any whole numbers a, b, and c with $(a \times c) \div (b \times c)$ a whole number,
$$a \div b = (a \times c) \div (b \times c).$$

8′: For any whole numbers a, b, and c with $(a \div c) \div (b \div c)$ a whole number,
$$(a \div c) \div (b \div c) = a \div b.$$

9′: For any whole numbers a, b, c, and d with $(a \div c) \times (b \div d)$ a whole number,
$$(a \times b) \div (c \times d) = (a \div c) \times (b \div d).$$

10′: For any whole numbers a, c, and d with $a \div (c \div d)$ a whole number,
$$a \div (c \div d) = (a \div c) \times d.$$

11′: For any whole numbers a, b, and c with $b \neq 0$,
$$(a \div b) \times c = (a \times c) \div b.$$

15. Let $n(A) = a$, $n(B) = b$, with A and B disjoint sets. Let $n(A') = a$, $n(B') = b$, with sets A' and B' disjoint. We must show that $n(A \cup B) = n(A' \cup B')$. This will hold provided we prove $A \cup B$ equivalent to $A' \cup B'$.

If $n(A) = a$ and $n(A') = a$, it follows that A and A' are both equivalent to the set $C = \{1, 2, \ldots, a\}$, the set of counting numbers from 1 up to and including a. We now show that A and A' are equivalent by showing they can be matched with a one-to-one correspondence. Let x be an element of A. As A and C are equivalent, there is a one-to-one correspondence between A and C.

Under this correspondence, x in A will be paired with some element in C, say y. Under the one-to-one correspondence between C and A', y will be paired with some element x' in A'. Under these two fixed matchings of A with C and C with A' we establish a one-to-one correspondence between A and A' that pairs x with x' (through y) and similarly provides pairings for all the other elements of A and A'. These pairings provide a one-to-one correspondence between A and A' so that A and A' are equivalent. Similarly we show that B and B' are equivalent. To show $A \cup B$ and $A' \cup B'$ equivalent, let x be a member of $A \cup B$. If x is in A, let x' of A' be its partner under the correspondence between A and A'. If x is in B, then let x' of B' be its partner under the correspondence between B and B'. As A and B are disjoint and A' and B' are also disjoint, the matching will be a one-to-one correspondence between $A \cup B$ and $A' \cup B'$. As $A \cup B$ and $A' \cup B'$ are now equivalent, both will have the same number assigned, $a + b$. Hence, it does not matter which sets A and B are chosen provided they are disjoint and

$$n(A) = a, \qquad n(B) = b.$$

Then

$$n(A \cup B) = a + b.$$

16. **a.** $(36 \times 47) + (47 \times 64) = (47 \times 36) + (47 \times 64)$
$= 47\,(36 + 64)$
$= 47 \times 100$
$= 4{,}700.$

b. $25 \times (36 \times 4) = 25 \times (4 \times 36)$
$= (25 \times 4) \times 36$
$= 100 \times 36$
$= 3{,}600.$

c. $575 - 298 = (575 + 2) - (298 + 2)$
$= 577 - 300$
$= 277.$

d. $575 + 298 = (573 + 2) + 298$
$= 573 + (2 + 298)$
$= 573 + 300$
$= 873.$

e. $575 \div 25 = (575 \times 4) \div (25 \times 4)$
$= 2{,}300 \div 100$
$= 23.$

17. If $a < b$, then for some whole number $w \neq 0$, $a + w = b$. But then $(a + w) + c = b + c$. Rearranging addends, we get

$$(a + c) + w = b + c.$$

Hence

$$a + c < b + c, \text{ as } w \neq 0.$$

18. As $a + w = b$, for $w \neq 0$,

$$(a + w) - c = b - c.$$

Then

$$(a - c) + w = b - c \text{ (see Exercise 5).}$$

Then

$$a - c < b - c, \text{ as } w \neq 0.$$

19. If $a < b$, then $a + w = b$ with $w \neq 0$. If $b < c$, then $b + w' = c$ with $w' \neq 0$. Hence

$$(a + w) + w' = c$$

or

$$a + (w + w') = c.$$

But $w \neq 0$, $w' \neq 0$, so that $w + w' \neq 0$. Hence $a < c$.

20. a. Let one odd number be $2a + 1$, the other odd number be $2b + 1$. Then their sum will be

$$\begin{aligned}(2a + 1) + (2b + 1) &= (2a + 2b) + (1 + 1)\\ &= 2(a + b) + (2 \times 1)\\ &= 2([a + b] + 1).\end{aligned}$$

As $(a + b) + 1$ is a whole number, $2([a + b] + 1)$ is an even number.

b. Let $2a$ be the even number, $2b + 1$ be the odd number. Then their sum will be

$$\begin{aligned}2a + (2b + 1) &= (2a + 2b) + 1\\ &= 2(a + b) + 1.\end{aligned}$$

As $a + b$ is a whole number, $2(a + b) + 1$ is an odd number, as it is 1 more than an even number.

c. Let the even number be $2a$ and c be the other factor. Then their product will be

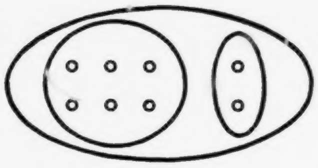
$$(2a)(c) = 2(ac).$$

But ac is a whole number. Hence $2(ac)$ is even.

d. Let $2a + 1$ be one odd number, $2b + 1$ be the other odd number. Then their product will be

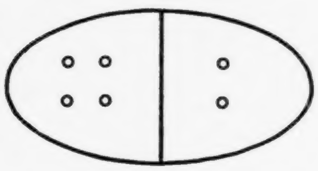
$$\begin{aligned}(2a + 1)(2b + 1) &= (2a + 1)(2b) + (2a + 1)(1)\\ &= \text{even number} + \text{odd number (see } \textbf{c}\text{)}.\\ &= \text{odd number (see } \textbf{b}\text{)}.\end{aligned}$$

21. a.



Count the elements in the union.

b.



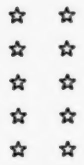
Count the elements in this set as

$4 + 2 = 6$

or

$6 - 2 = 4$.

c.



Count the elements in this array.

d. As $3 \times 2 = 6$, $6 \div 2 = 3$.

Exercise Set 3, pp. 163-64

1.

NUMBER PAIR	+ SUM	− DIFFERENCE	× PRODUCT	÷ QUOTIENT	v AVERAGE
(12, 4)	16	8	48	3	8
(44,4)	48	40	176	11	24
(4, 4)	8	0	16	1	4
(4, 12)	16		48		8
(2, 0)	2	2	0		1
(0, 2)	2		0	0	1
(0, 0)	0	0	0		0

2.

NUMBER	DOUBLE	TRIPLE	SQUARE	SUCCESSOR	PREDECESSOR
6	2×6 or 12	3×6 or 18	6×6 or 36	$6 + 1$ or 7	$6 - 1$ or 5
2	2×2 or 4	3×2 or 6	2×2 or 4	$2 + 1$ or 3	$2 - 1$ or 1
7	2×7 or 14	3×7 or 21	7×7 or 49	$7 + 1$ or 8	$7 - 1$ or 6
1	2×1 or 2	3×1 or 3	1×1 or 1	$1 + 1$ or 2	$1 - 1$ or 0
0	2×0 or 0	3×0 or 0	0×0 or 0	$0 + 1$ or 1	
10	2×10 or 20	3×10 or 30	10×10 or 100	$10 + 1$ or 11	$10 - 1$ or 9

Exercise Set 4, p. 164

1. 5 is less than 7 because $5 + 2 = 7$.

2. "a is less than b" means "there is a number $c \neq 0$ such that $a + c = b$."

3. "$a - b$" is meaningful in whole numbers whenever a is not less than b.

4. If $a < b$, then for some $c \neq 0$, $a + c = b$. But then

$$(a + c) + 1 = b + 1$$

or

$$(a + 1) + c = b + 1 \text{ (rearranging addends).}$$

Therefore,

$$a + 1 < b + 1, \text{ as } c \neq 0.$$

5. By pairing a pencil with a crayon until one collection or both are exhausted. If one is exhausted before the other, the exhausted collection had fewer elements. If both collections were exhausted at the same time, the collections had the same number of elements at the beginning.

6. $a = b$, $a < b$, $b < a$.

GLOSSARY

GLOSSARY

The particular words and symbols in this glossary have been selected to help clear up any possible misunderstanding. Usually a description rather than a precise definition is given. Moreover, not all the meanings are given but only those needed for the text materials. Examples are provided to clarify meanings still further.

Abacus. An ancient device (still used today) for computing. A common type consists of a frame with parallel rods. The rods are usually matched with the ones place, the tens place, the hundreds place, and so on. Movable counters along the rods record numbers and are used to carry out computation.

Addend. One of the numbers added to determine a sum. When a pair of numbers is associated with a sum under addition, each number of the pair is called an addend of the sum. In the sentence $6 + 7 = 13$, the numbers 6 and 7 are addends. In more general terms, $a + b$ is the sum of its addends a and b. In the sentence $6 + \square = 13$, one of the addends is "missing." See **Missing addend.**

Addition. With every pair of numbers a and b, addition associates the sum $a + b$. For example, with the pair 13 and 6, addition associates $13 + 6$, or 19. The sum $a + b$ may be determined in the following way:

If A and B are disjoint sets such that $n(A) = a$ *and* $n(B) = b$, *then*

$$a + b = n(A \cup B).$$

Addition property of zero. For every whole number b, $b + 0 = b$ and $0 + b = b$. Informally stated, the sum of every whole number and zero is the given whole number. See **Identity element for addition.**

Additive property of numeration systems. Each symbol in a numeral stands for a number. The sum of these numbers is the value of the numeral. In XXIII, the individual symbols stand for 10, 10, 1, 1, and 1. Because of the additive property, the number represented by XXIII is $10 + 10 + 1 + 1 + 1$, or 23.

Algorithm (algorism). A systematic, step-by-step procedure for reaching some goal. An algorithm for a subtraction computation is a systematic procedure for obtaining a standard name for a difference. See **Compute.**

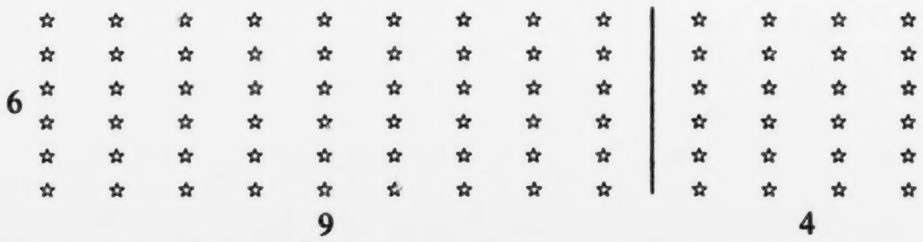
One of the possible algorithms for computing 624 − 397 yields the following steps:

(a)
$$\begin{array}{r} 6\,2\,{}^{1}4 \\ -3\,9_{1}7 \\ \hline 7 \end{array}$$

(b)
$$\begin{array}{r} 6\,{}^{1}2\,{}^{1}4 \\ -3_{1}9_{1}7 \\ \hline 2\,7 \end{array}$$

(c)
$$\begin{array}{r} 6\,{}^{1}2\,{}^{1}4 \\ -3_{1}9_{1}7 \\ \hline 2\,2\,7 \end{array}$$

Array, rectangular. A rectangular arrangement of objects in rows and columns. The array below, viewed both as a whole and as split into two parts, illustrates the distributive property.



The large array has 6 rows and 13 columns. We say that it is a 6-by-13 or 6 x 13 array. The 6 x 13 array is shown partitioned into two arrays, a 6 x 9 array and a 6 x 4 array, showing that $6 \times (9 + 4) = (6 \times 9) + (6 \times 4)$.

Associative property of addition. (Also called the **grouping property of addition.**) Whenever a, b, and c are whole numbers, $a + (b + c) = (a + b) + c$. That is, when numbers are added, the grouping of the numbers does not affect the sum. An instance of this property is the fact that $6 + (9 + 4) = (6 + 9) + 4$. Subtraction, on the other hand, is not associative. A single exception, although there are many, suffices to show this: $(8 - 5) - 2 \neq 8 - (5 - 2)$.

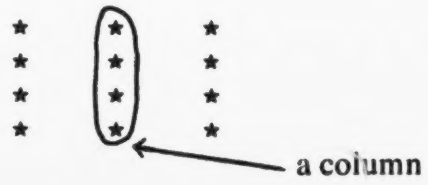
Associative property of multiplication. (Also called the **grouping property of multiplication.**) Whenever a, b, and c are whole numbers, $a \times (b \times c) = (a \times b) \times c$. That is, when numbers are multiplied, the grouping of the factors does not affect the product. An instance of this property is the fact that $3 \times (7 \times 5) = (3 \times 7) \times 5$. Division, on the other hand, is not associative. A single exception, although there are many, suffices to show this: $(8 \div 4) \div 2 \neq 8 \div (4 \div 2)$.

Base. A number used as a repeated factor. In the expression $10^3 = 10 \times 10 \times 10$, for example, the base is shown to be 10. We refer to 3 as the exponent. See **Exponent** and **Factor.** The symbol "10" is a name for the number ten in our Hindu-Arabic decimal numeration system, but it is not a name for the number ten in systems with other bases.

Base-sixty system. A system of writing numerals designed to represent ones, sixties, sixty sixties, and so on.

Base-ten system. A system of writing numerals based upon ones, tens, ten tens, and so on. The Egyptian system of numeration is a base-ten system, as is the Hindu-Arabic system.

Column. A vertical line of objects in an array. The array below has three columns.



Commutative property of addition. (Also called the **order property of addition.**) Whenever a and b are whole numbers, $a + b = b + a$. That is, when two numbers are added, the order in which the numbers are added or the order of the addends does not affect the sum. An instance of this property is the fact that $9 + 4 = 4 + 9$.

Commutative property of multiplication. (Also called the **order property of multiplication.**) Whenever a and b are whole numbers, $a \times b = b \times a$. That is, when two numbers are multiplied, the order in which they are multiplied or the order of the factors does not affect the product. An instance of this property is the fact that $6 \times 14 = 14 \times 6$.

Computation. A process for finding the standard numeral for a sum, a product, etc.; a process for finding a standard name.

Compute. To find a standard numeral for a sum, a product, etc. To compute the sum of 34 and 8 means to find the standard numeral for $34 + 8$, namely "42." To find a standard name.

Correspondence. A pairing of the members of two sets whereby each member of the first set is paired with a member of the second set, and never with more than one member of the second set. See also **One-to-one correspondence** and **Operation.**

Counting. The process of pairing the elements of a set with the counting numbers taken "one after another" in order of "size" and starting with 1. If this process stops, the last counting number used is the number of elements in the set being counted. When this happens, the set is said to be a *finite set*, and the number associated with the set a *finite number*. Every whole number is a finite number.

Counting number. Any whole number other than 0. (Some authors include 0 among the counting numbers.)

Cross product. The cross product of a pair of sets is the set of all ordered pairs whose first element is from the first set and whose second element is from the second set. The cross product of $\{a, b\}$ and $\{x, y, z\}$ is $\{(a, x), (a, y), (a, z), (b, x), (b, y), (b, z)\}$. See **Symbol, X.**

Decimal. Pertaining to ten (from the Latin word *decima*, meaning "tithe" or "a tenth part.")

Decimal numeration system. A system for naming numbers based on tens. See **Hindu-Arabic system of numeration.**

Difference. A number assigned to certain pairs of whole numbers by subtraction. $18 - 11$, or 7, is the difference of 18 and 11. See **Missing addend.** $a - b$ is the difference of a and b, provided a is not less than b. If a set A has a elements and one of its subsets, B, has b elements, then the number of elements in A but not in B is $a - b$. Alternately, the difference $a - b$ is the missing addend in $\square + b = a$.

Digits. The basic symbols in a numeration system. In the Hindu-Arabic system the digits are 0, 1, 2, 3, 4, 5, 6, 7, 8, 9.

Disjoint sets. Two sets are disjoint if they have no elements in common. If a, b, c, d, and e are distinct objects, then the sets $\{a, b\}$ and $\{c, d, e\}$, are disjoint; but the sets $\{a, b\}$ and $\{b, c\}$ are not disjoint.

Distributive property, or **distributive property of multiplication over addition.** Whenever a, b, and c are whole numbers,

$$a \times (b + c) = (a \times b) + (a \times c).$$

An instance of this property is the fact that

$$13 \times (8 + 7) = (13 \times 8) + (13 \times 7).$$

Because multiplication is commutative, the distributive property may also be written in the form $(b + c) \times a = (b \times a) + (c \times a)$.

Dividend. In the sentence $a \div b = q$, the number a is called the dividend. The number a is also called the dividend in the sentence $a = (q \times b) + r$, with $r < b$. In the sentence $15 \div 3 = 5$, the number 15 is the dividend. In the sentence $15 = (2 \times 7) + 1$, the number 15 is again the dividend.

Division. With certain pairs of whole numbers, a and b, division associates the quotient, $a \div b$. Division assigns to certain pairs of whole numbers a and b a unique whole number $a \div b$. Such a unique whole number $a \div b$ exists provided $b \neq 0$ and there is a number c such that $c \times b = a$. For example, $51 \div 3 = 17$ because $17 \times 3 = 51$. (Of course $3 \neq 0$.) Division assigns $72 \div 9$, or 8, to the pair 72 and 9. The standard name for $a \div b$ can be obtained in three ways:

1. If a set of a elements can be partitioned into disjoint subsets of b elements each, then the number of subsets thus formed is $a \div b$.
2. If an array has a elements and b rows, then the number of columns of the array is $a \div b$. If an array has a elements and b columns, then the number of rows is $a \div b$.
3. If a and b are whole numbers, the whole number that correctly completes the sentence $b \times \square = a$, or $\square \times b = a$, is $a \div b$, provided there is exactly one such whole number.

Division by zero. Division by zero has no meaning. The expressions $5 \div 0$, $18 \div 0$, $0 \div 0$, $1 \div 0$, etc., do not name numbers. Division by 0 is meaningless because there are no whole numbers that fit sentences like the ones at the top of the following page,

$$0 \times \square = 5, \qquad \square \times 0 = 18, \qquad 0 \times \square = 1, \qquad \text{etc.,}$$

and because every whole number fits the sentence $0 \times \square = 0$.

Division with a remainder assigns a quotient and a remainder to a pair of whole numbers. If a and b are whole numbers ($b \neq 0$), then there are whole numbers, a quotient q and a remainder r (with $r < b$), which satisfy the equation

$$a = (b \times q) + r.$$

If $a = 23$, $b = 4$, division with a remainder determines $q = 5$ and $r = 3$.

$$23 = (4 \times 5) + 3.$$

Divisor. In the sentence $a \div b = q$, the number b is called the divisor. The number b is also called the divisor in the sentence $a = (q \times b) + r$. For example, in the sentence $15 \div 3 = 5$, the number 3 is the divisor; in the sentence $15 = (3 \times 4) + 3$, or $15 = (\square \times 4) + \triangle$, the number 4 is the divisor.

Element of a set. Each object in any nonempty set of objects is an element of the given set. For example, the set {New York, California, Michigan} has three elements: New York, California, and Michigan.

Empty set. The set that has no elements, the null set. Often designated by either the symbol { } or $\emptyset$. Examples: the set of people 30 feet tall; the set of female presidents of the United States.

Equal sign. See **Symbol.**

Equivalent. If there is a one-to-one correspondence between two sets, then the sets are said to be equivalent. Sets that are equivalent are assigned the same number. Sets that are not equivalent are not assigned the same number. Examples of equivalent sets are $\{a, b\}$ and {blue, green}.

Expanded form. An expanded form of a decimal numeral is a numeral that shows explicitly the place value of the digits in a decimal numeral. Expanded forms of the numeral 456 include:

$$400 + 50 + 6$$
$$(4 \times 100) + (5 \times 10) + (6 \times 1)$$
$$(4 \times 10^2) + (5 \times 10) + (6 \times 1)$$
$$(4 \times 10^2) + (5 \times 10^1) + (6 \times 10^0)$$

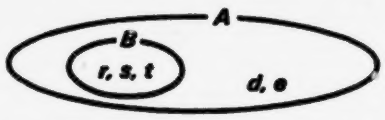
Exponent. A number used to indicate a repeated factor. The repeated factor is called the base. In 10^2, 2 is the exponent and 10 is the base. 10^2 means 10×10. In 10^3, 3 is the exponent and 10 is the base. 10^3 means $10 \times 10 \times 10$. See **Base.**

Factor. One of the numbers multiplied to determine a product. When a pair of numbers is associated with a product under multiplication, each number of the pair is called a factor of the product. In the sentence $3 \times 4 = 12$, 3 and 4 are factors of 12. In general terms, if $a \times b = c$, a and b are factors of c. In the sentence $3 \times \square = 12$, one of the factors is "missing." See **Missing factor.**

Family. A collection of sets, every two of which are equivalent.

Greater than

1. Whole number *a* is greater than whole number *b* if there is a whole number *c*, other than 0, such that $a = b + c$. For example, 5 is greater than 3 because there is the whole number 2 such that $5 = 3 + 2$.
2. Whole number *a* is greater than whole number *b* if there are two sets *A* and *B* such that set *A* contains all the elements of set *B*, *A* has at least one element not in *B*, *a* is the number of elements in set *A*, and *b* is the number of elements in set *B*. For example, 5 is greater than 3 because (see diagram) $n(A) = 5$, $n(B) = 3$, and set *A* has elements *d* and *e* which are not in set *B*.



3. Let whole number *a* be the number of elements in set *A*. Let whole number *b* be the number of elements in set *B*. We say that *a* is "greater than" *b* if and only if set *B* can be matched with a proper subset of set *A*. (In this case, we say that set *A* has "more" elements than set *B*.)

Grouping property of addition. See **Associative property of addition.**

Grouping property of multiplication. See **Associative property of multiplication.**

Hindu-Arabic system of numeration. Our decimal system for naming numbers. All whole numbers can be expressed using ten digits and the idea of place value.

Identity element for addition. The number 0 is the identity element for addition of whole numbers because whenever *b* is a whole number, $b + 0 = b$ and $0 + b = b$. That is, when 0 is an addend, the sum is the same number as the other addend. The number 0 is sometimes called the *neutral* element for addition. For example, $4 + 0 = 4$; $0 + 56 = 56$; etc.

Identity element for multiplication. The number 1 is the identity element for multiplication of whole numbers because whenever *b* is a whole number, $b \times 1 = b$ and $1 \times b = b$. That is, when 1 is a factor, the product is the same as the other factor. The number 1 is sometimes called the *neutral* element for multiplication. For example, $4 \times 1 = 4$; $1 \times 56 = 56$; etc.

Known addend. In a sentence such as $\square + 8 = 14$, 8 is the known addend, or given addend. See **Missing addend.**

Known factor. In a sentence such as $\square \times 3 = 12$, 3 is the known factor, or given factor. See **Missing factor.**

Less than. *a* is less than *b* means *b* is greater than *a*. See **Greater than.**

Match. See **One-to-one correspondence.**

Member of a set. In any nonempty set of objects each object is a member of the set. Synonymous with **element of a set**. For example, the set consisting of the elements *a*, *b*, *c* has for its members *a*, *b*, *c*. It follows that *a* is a member of this set, *b* is a member of this set, and *c* is a member of this set. The members of the set {(3, 7), 9} are (3, 7) and 9. The members of a set may be ordered pairs as well as numbers.

Minus. The name for the subtraction symbol, "−." See under **Symbol.**

Missing addend. In a sentence such as $8 + \square = 12$, one of the addends is not given, or is "missing." The □, called a frame, provides a place in which to name the missing addend. Determination of the missing addend in $8 + \square = 12$ corresponds to subtracting 8 from 12. That is, since $8 + 4 = 12$, $4 = 12 - 8$.

Missing factor. In a sentence such as $\square \times 8 = 40$, one of the factors is not given, or is "missing." The □, called a frame, provides a place in which to name the missing factor. Determination of the missing factor in $\square \times 8 = 40$ corresponds to dividing 40 by 8. That is, since $5 \times 8 = 40$, $5 = 40 \div 8$.

Multiple. A whole number *a* is a multiple of a whole number *b* if there is a whole number *c* such that $a = b \times c$. For example, 30 is a multiple of 10 because $30 = 10 \times 3$. 28 is a multiple of 7 because $28 = 7 \times 4$. The multiples of 10 are 0, 10, 20, 30, 40, ... The set of multiples of a nonzero number is an infinite set.

Multiplication. With every pair of whole numbers *a* and *b* multiplication associates the product $a \times b$. For example, with the pair 7 and 9 multiplication assigns the product 7×9, or 63. The product $a \times b$ can be computed in the following ways:

1. If set *A* contains *a* elements and set *B* contains *b* elements, then $a \times b = n(A \times B)$, the number of elements in the cross-product set, $A \times B$.
2. Choose *a* sets, disjoint from each other, with *b* elements in each of the *a* sets. Then $a \times b$ is the number of elements in the union of these sets.
3. The number of elements in an *a*-by-*b* array is $a \times b$.
4. On a number line, the product $a \times b$ is the number of units in a single "jump" that covers the same distance as *a* jumps with *b* units in each jump.

Multiplication property of one. For any whole number *a*, $a \times 1 = a$ and $1 \times a = a$. Informally stated, the product of any whole number and 1 is that whole number. See **One in division.**

Multiplication property of zero. For any whole number *a*, $a \times 0 = 0$ and $0 \times a = 0$. Informally stated, the product of any whole number and 0 is 0. See **Division by zero.**

$n(A)$ is the number of elements in set *A*. See under **Symbol.**

Natural number. Each of the numbers 1, 2, 3, 4, 5, ... ; any whole number except 0. (Some authors include 0 as a natural number, but we do not.)

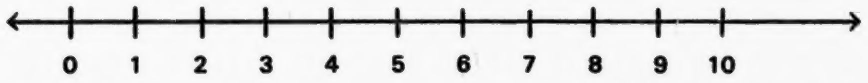
Neutral element. Same meaning as identity element. See **Identity element for addition** and **Identity element for multiplication.**

Notation, system of. See **Numeration system.**

Null set. See **Empty set.**

Number. A basic idea which is associated with a set. This association of a number with a set is made in such a way that two equivalent sets are associated with the same number while two nonequivalent sets are associated with different numbers. See **Counting number, Natural number,** and **Whole number.**

Number line. A drawing of a line (with arrows to indicate unlimited length) on which a unit length has been selected and marked off consecutively beginning at any fixed place and moving to the right. The marks are labeled in order "0," "1," "2," "3," "4," "5," and so on. The drawing below is an example of a number line.

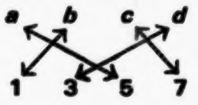


Numeral. Mark or name for a number; any symbol that names a number. For example, some numbers for the number five are "V," "4 + 1," "five," "5."

Numeration system. A scheme for naming numbers. Any organized system of using words or marks to denote numbers. Examples: decimal numeration system, Roman numeration system, Egyptian numeration system.

One in division. For any whole number a, $a \div 1 = a$ and, if $a \neq 0$, $a \div a = 1$. Informally stated, any whole number divided by 1 is that whole number, and any whole number (except 0) divided by itself is 1.

One-to-one correspondence between two sets. A pairing of the members of the two sets, not necessarily different sets, so that each pair contains exactly one member from each set, and each element of each set is in exactly one pair. For example, one-to-one correspondence between the sets $\{a, b, c, d\}$ and $\{1, 3, 5, 7\}$ is shown by the accompanying diagram. This correspondence can also be shown by listing the pairs: $(a, 5)$, $(b, 1)$, $(c, 7)$, $(d, 3)$.



Operation. A set of associations for elements of two sets, pairing each member of the first set with a member of the second set, but never pairing the same member of the first set with more than one member of the second set. In a binary operation the elements of the first set are ordered pairs. According to this definition, addition, subtraction, multiplication, and division are binary operations. An operation is a correspondence. [More on page following.]

Mathematicians generally consider a binary operation on, say, set A, to be more restricted: A binary operation on set A is a correspondence between $A \times A$ and A such that *every* member of $A \times A$ has a partner in set A. Under this restricted definition for whole numbers, addition and multiplication are still binary operations while subtraction and division are not.

Order property of addition. See **Commutative property of addition.**

Order property of multiplication. See **Commutative property of multiplication.**

Ordered pair. Two objects considered together where one of the objects is first in the pair and the other is second in the pair. The ordered pair of numbers (4, 7) is different from the ordered pair (7, 4). In an ordered pair the first and second elements (also called components) may be the same, as in (7, 7).

Ordered set. The only ordered set to which we make reference in this text is the ordered set of counting numbers {1, 2, 3 ,4 ,5, ... }. This particular listing in the braces means that its members are assigned specific positions in the ordering; namely, 1 is the first number, 2 is the second number, 3 is the third, etc. The ordering of the counting numbers used here is according to "size." Each number is 1 less than its successor. This particular ordering is essential for counting.

Pair. See **Ordered pair.**

Partial quotient. When a quotient has been computed as a sum, each addend of this sum is called a partial quotient. For instance, in computing 8,972 ÷ 24, we obtain the quotient 300 + 70 + 3. Each of the numbers (300 or 70 or 3) is a partial quotient.

$$\begin{array}{r|r} 24\,\overline{)\,8{,}972} & \\ \underline{7{,}200} & 300 \\ 1{,}772 & \\ \underline{1{,}680} & 70 \\ 92 & \\ \underline{72} & \underline{3} \\ r = \quad 20 & 373 = q \end{array}$$

Partition. To partition a set is to split up the set into nonempty disjoint subsets so that every element in the set is in exactly one of the subsets. See **Division** (1).

Place value. The number assigned to each position occupied by a digit in a standard numeral. In the standard numeral "289," the "2" occupies the position to which the value 100 is assigned. We say the "2" is in the hundreds place. The place value of "2" in "289" is 100.

Plus. A name for the symbol "+." See under **Symbol.**

Positional value of digits in a numeral. See **Place value.**

Powers of ten. In this book, "powers of ten" refers to the numbers 1, 10, 100, 1,000, etc. These numbers are also expressed as 10^0, 10^1, 10^2, 10^3, etc.

Product. With every pair of whole numbers a and b multiplication associates the product "$a \times b$." The product of whole numbers a and b, denoted by $a \times b$, is the number of elements in a cross-product set $A \times B$, where $n(A) = a$ and $n(B) = b$. Alternately, $a \times b$ is the number of elements in the union of a sets, disjoint from each other, with b elements in each. Finally, $a \times b$, the product of a and b, is the number of elements in an array having a rows and b columns. Example: 8 is the product of 4 and 2. See **Multiplication**.

Proper subset. Set A is a proper subset of set B if A is a subset of B while B contains at least one element which is not a member of A. Example: If a, b, c, and y are distinct elements, $\{a, b\}$ is a proper subset of $\{a, b, c, y\}$.

Quotient. A number assigned to certain pairs of whole numbers by division. In the sentence $a \div b = q$, the number q is called the quotient of a and b. When we try to compute $a \div b$, the unique whole number q for which $a = (q \times b) + r$ with $r < b$ is also called the quotient. Examples: The quotient of 15 and 3 is $15 \div 3$, or 5. In $17 = (2 \times 7) + 3$, the number 2 is the quotient when we regard 7 as the divisor.

Remainder. When we try to compute $a \div b$, the unique whole number r less than b for which $a = (q \times b) + r$ is called the remainder. For example, in $15 = (2 \times 7) + 1$, the number 1 is the remainder.

Repeated addition. If m and n are whole numbers,

$$m \times n = \underbrace{n + n + n + \ldots + n + n.}_{m \text{ addends.}}$$

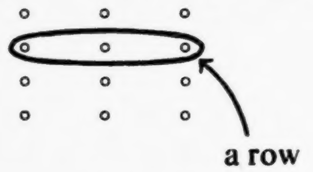
Thus, for example,

$$3 \times 4 = 4 + 4 + 4.$$

If $m = 1$, the right side is interpreted to mean n. If $m = 0$, the right side is taken to mean 0.

Repetitive property. A numeration system has this property provided, when any of its basic symbols is repeated in a numeral, it has the same particular value regardless of its position. In the Egyptian system, each basic symbol in "∩ ∩ ∩" has the same value, 10. In our system, each digit in "333" represents a different number according to its position in the numeral, so our decimal numeration system does not have the repetitive property.

Row. A horizontal line of objects in an array. The array below has four rows.



Set. Collection or group or aggregate of objects that may be concrete or abstract, similar or dissimilar. One would usually like to be able to decide if any particular object is or is not a member of the set. Mathematicians usually do not define "set."

Standard form of a numeral (Standard numeral, standard name for a number). In the Hindu-Arabic system, a numeral consisting of digits only without any sign of operation. The symbols "0," "1," "2," "3," "4," "5," ... "11," "12," "13," ... . See **Expanded form.** The standard name for 2 + 3 is "5."

Standard numeral. See **Standard form of a numeral.**

Subset. Set A is a subset of set B if every member of A is also a member of B. Alternately, set A is a subset of set B if every element not in B is also not in A. As a special case, A may be the entire set B itself. As another special case, A may be the empty set; that is, A may have no elements. Thus, if set A is identical to set B, or if A is the empty set, set A is a subset of set B.

Subtraction. With every pair of whole numbers a and b, provided a is not less than b, subtraction assigns the difference of a and b, denoted by $a - b$. For example, the difference of 8 and 2 is $8 - 2$, or 6.

Successor. If n is a whole number, then $n + 1$ is the successor of n. For example, the successor of 0 is 1; the successor of 8 is 9; etc.

Sum. With every pair of numbers a and b addition associates the sum $a + b$. The sum of whole numbers a and b, denoted by $a + b$, is the number of elements in the union of sets A and B provided that $n(A) = a$, $n(B) = b$, and sets A and B are disjoint. For example, $4 + 2$, or 6, is the sum of 4 and 2. See **Addition.**

Symbol. A mark, a collection of marks, or an expression that is used to communicate an idea. For example, numerals are symbols for numbers. Some special mathematical symbols follow:

{ } Braces. Sometimes called curly brackets. Consist of two symbols used to enclose the names of members of a set or a description of the members of a set, as $\{a, b, c\}$ and {even numbers}. If nothing appears between the braces, then the set has no members and is the empty set, designated by { }.

= Equal sign. The symbol is used between two expressions to assert that the expressions name the same thing and, in particular, when referring to numbers, name the same number. For example, $3 + 3 = 4 + 2$; $\{a, b\} = \{b, a\}$.

$n(A)$ An abbreviation for any one of the following synonymous expressions:

a) the number of elements in set A
b) the number associated with set A
c) the number property of set A
d) the number of set A
e) the cardinal number of set A

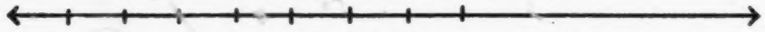
... Three dots, as in 1, 2, 3, 4, 5, ..., signify that the indicated pattern (in this case, of adding one) is to continue indefinitely.

+	The plus sign, a symbol of addition. The symbol "$a + b$" (read "a plus b") names the sum of numbers a and b. See **Addition.**
∪	A symbol for union. $A \cup B$ (read "A union B") names the union of sets A and B. $A \cup B \cup C$ means the union of sets A, B, and C.
>	means "is greater than." For example, $5 > 3$ means 5 is greater than 3.
<	means "is less than." For example, $3 < 5$ means 3 is less than 5.
–	The minus sign, the symbol for subtraction. $a - b$ (read "a minus b") names the difference of a and b, that is, the missing addend in the sentence $\square + b = a$. See **Subtraction.**
X	The symbol for cross product. $A \times B$ (read "A cross B") names the cross product of sets A and B. See **Cross product.**
×	The "times sign," the symbol for multiplication. $a \times b$ (read "a times b") names the product of numbers a and b. See **Multiplication.**
÷	The **symbol for division.** $a \div b$ (read "a divided by b") names the result of dividing a by b, that is, the missing factor in $\square \times b = a$ or $b \times \square = a$, that is, the quotient of a and b. See **Division.**
≠	Means "is not equal to," "does not equal." The symbol is used between two expressions to assert that the expressions do not name the same thing. For example, $5 + 1 \neq 8$ asserts that $5 + 1$ and 8 are different numbers.
□	A frame for entering a symbol. Examples: Compute the missing number: $\square = 14 + 2$. Determine the missing operation: $2 \square 3 = 6$. When the same frame is repeated in a sentence, the same symbol must be used. If different frames are used, the symbols need not be different.

Union. The union of two sets is the set consisting of all the elements that are in either or both of the two sets. The union of $\{x, y\}$ and $\{y, z, w\}$ is $\{w, x, y, z\}$. See **Symbol,** ∪. If $A = \{x, y\}$, $B = \{y, z, w\}$, then the union of A and B is denoted by $A \cup B$. Thus $A \cup B = \{w, x, y, z\}$. The union of two sets is the set that contains all the elements of each set and no others. In more general terms, the union of any collection of sets is the set consisting of all those elements that are members of at least one of the sets in the given collection.

Unit. In this book, the word "unit" is used in reference to a representation of a number line. Any length we wish to select is used as a basic length to be marked off consecutively on the illustrated line.

For example, we might choose as our unit the segment illustrated by ⊢—⊣ . Then we mark consecutively on the drawing of the number line as many of these lengths as we want.



On the above representation of the number line, we have marked off 7 of the selected units.

Whole number. One of the numbers 0, 1, 2, 3, ... The set of whole numbers consists of 0 and the counting numbers.